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Nonlinear dynamics of filaments in free space and fluids

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Nonlinear dynamics of filaments in free space and fluids

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Abstract

The purpose of this paper is to study a straight rod, held at both ends, with a known twist and tension or compression. We study the stability of this steady state when the system is dominated either by inertia or drag. In order to do this, we first replicate the work of Goriely and Tabor to look at the case with inertia, without drag. After conducting the analysis for that case, we then apply their framework to perform a linear stability analysis of a model that is without inertia, but with hydrodynamic drag. Our motivation is the study of locomotion of *C.elegans* and other long, slender organisms such as bacterial flagella, cilia, and DNA.

CONTENTS

I. ACKNOWLEDGEMENTS

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II. INTRODUCTION

We analyze a straight rod with twist under tension or compression using the methods of Goriely and Tabor [2]. In our application, we consider the rod held under compression or tension in a viscous fluid where the governing external force is no longer inertia, but rather hydrodynamic drag. Intuition tells us that if the rod is twisted enough, it should buckle into a helix, or bend back onto itself. So we expect that if a given rod or filament is twisted, but held at both ends, there is a critical twist after which the straight configuration becomes unstable and the stability should be related to both the amount of tension or compression and the twist along with the material parameters of and external damping applied to the rod.

The steady state is stable if the perturbations decay, meaning after perturbation, the rod returns to its steady state (or approaches the steady state configuration). It is unstable if the perturbations grow and the rod tends toward another configuration, like a helical coil [2]. To study this stability, or lack thereof, in our situation of interest, we first looked at Goriely and Tabor's analysis of a straight rod under twist and tension with inertia, without drag. To understand this problem, we reproduce their work by linearizing the balance of linear and angular momentum for the case with inertia, without drag in Sections IV and V. We are then able to modify their linearized system to account for our model of interest which includes drag, as resistive force theory, but not inertia in Section VI [4]. Understanding the stability of a thin object subject to two different types of forces, internal inertia and external drag, can further our understanding of the alternate effects of changing the inertia and drag has on the stability.

III. THE KIRCHHOFF ROD AND SLENDER BODY THEORY

If an object is much longer in one dimension than in the other two, such as a rod or worm-like organism, we may represent it as a one-dimensional filament, as shown in Figure 1.

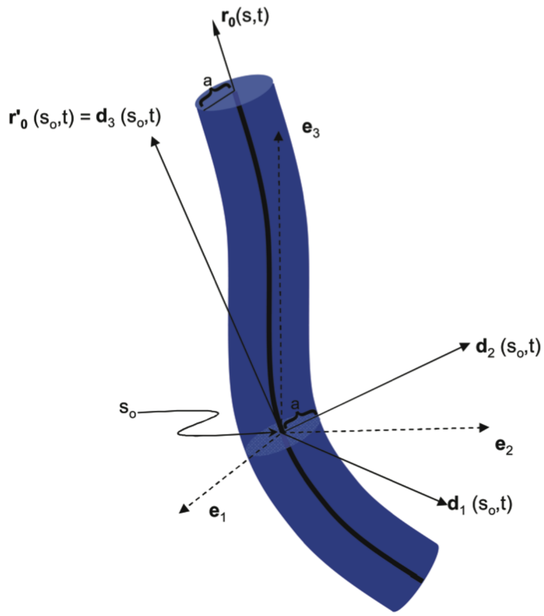

FIG. 1. A short segment of the rod depicting the Kirchhoff frame, $\vec{\mathbf{d}}_i$, and the centerline, $\mathbf{r}_0(s,t)$, where s is arc length, t is time, and the $\vec{\mathbf{e}}_i$ are a fixed coordinate frame. The $\vec{\mathbf{d}}_i$ are a local material frame.

Figure 1 shows a segment of a rod with $\mathbf{r}_0(s,t)$ as the centerline and three local basis vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2$, and $\vec{\mathbf{d}}_3$. From this, $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$ are usually chosen to be in some natural material direction, like a normal and binormal vector respectively. The arc length derivative of the centerline is not only in the direction of the tangent vector $\vec{\mathbf{d}}_3$, but is, in fact equal to $\vec{\mathbf{d}}_3$, meaning that our rod is assumed to be inextensible. For the purpose of the paper we let $\frac{\partial}{\partial s} = ()'$, $\frac{\partial}{\partial t} = \dot{()}$, and indices $i, j, k \in \{1, 2, 3\}$. A similar theory exists for rods with small extensions, but biologically speaking, extension of an organism or flagella is not physically reasonable so is excluded in our studies. Looking at $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$, they are perpendicular to the centerline, so lie in the undeformed circular cross section of the rod. In this work, we are only concerned with a circular cross section but Kirchhoff theory is not restricted to such in general. Combined, the vectors $\vec{\mathbf{d}}_i$ are the local orthonormal basis of the Kirchhoff rod. All dependent variables shown are functions of the independent variables arc length, s, and time, t. The vectors $\vec{\kappa}$ and $\vec{\omega}$ are defined by

$$\begin{aligned} \vec{\mathbf{d}}_i' &= \vec{\kappa} \times \vec{\mathbf{d}}_i, \\ \dot{\vec{\mathbf{d}}}_i &= \vec{\omega} \times \vec{\mathbf{d}}_i. \end{aligned} \tag{1}$$

Here $\vec{\kappa}$ is essentially the bend and twist vector with respect to space while $\vec{\omega}$ is the bend and twist vector with respect to time. This means that each κ_i and ω_i represents rotation around the respective $\vec{\mathbf{d}}_i$ basis vector in either space or time. So κ_1, κ_2, ω_1, and ω_2 represent bending, while κ_3 and ω_3 represent twist. The primary motivation for using the Kirchhoff rod model is that it exploits the fact that the length, L, of the rod is much larger than the radius, a, while also using a very intuitive material frame, $\{\vec{\mathbf{d}}_i\}$. As a result, the rod can be treated as a one-dimensional curve in three-dimensional space.

IV. BALANCE OF LINEAR AND ANGULAR MOMENTUM WITHOUT DRAG

IV.1. Balance of Linear Momentum

The balance of linear momentum is given by

$$\vec{\mathbf{F}}'' = \rho A \ddot{\vec{\mathbf{d}}}_3,$$

where A is the area of a circular cross-section and ρ is the density. This equation balances the forces the cross-sections exert on each other. The balance of linear forces here is given by Newton's law, $F = ma$, so the total internal and external forces (the left-hand side) must sum to the mass (ρA) times the acceleration of a point on the filament ($\ddot{\vec{\mathbf{d}}}_3$). Here, there are no external forces on the individual cross-sections so the internal forces are simply the resultant force of one cross-section on another ($\mathbf{F}''$) [2]. We nondimensionalize with $\vec{\mathbf{F}} = F_0\hat{\mathbf{F}}$, and by choosing $F_0 = \rho A$, our balance of linear momentum becomes

$$\vec{\mathbf{F}}'' = \ddot{\vec{\mathbf{d}}}_3, \tag{2}$$

by dropping the hat notation as in the work by Goriely and Tabor.

IV.2. Balance of Angular Momentum with a Constitutive Relation

From [2], the balance of angular momentum in the Kirchhoff frame for a one-dimensional filament with twist and bend is given by

$$\rho I\left(\vec{\mathbf{d}}_1 \times \ddot{\vec{\mathbf{d}}}_1 + \vec{\mathbf{d}}_2 \times \ddot{\vec{\mathbf{d}}}_2\right) = \vec{\mathbf{M}}' + \vec{\mathbf{d}}_3 \times \vec{\mathbf{F}}. \tag{3}$$

Equation (3) is the balance of angular momentum, so if we twist one cross-section of the rod, we can see the effect the twist has on the nearby cross-sections of the rod. The constitutive relation for a linearly elastic Kirchhoff rod is given by

$$\vec{\mathbf{M}} = IE\kappa_\beta\vec{\mathbf{d}}_\beta + 2I\mu\kappa_3\vec{\mathbf{d}}_3. \tag{4}$$

Here, we use the convention of summation over repeated indices where $\beta \in \{1,2\}$ and $i,j,k \in \{1,2,3\}$. Assuming the rod is naturally straight, untwisted, and linearly elastic yields Equation (4). Here, I is the moment of inertia about a radial cross-section, $\vec{\mathbf{M}}'$ is the internal resultant torque, $\vec{\mathbf{d}}_3 \times \vec{\mathbf{F}}$ is the torque that results from the internal force, $\vec{\mathbf{F}}$ is the internal force acting on the cross-section, E is the Young's

Modulus, and μ is the shear modulus of the filament [2]. We nondimensionalize the balance of angular momentum by choosing $M_0 = \frac{EI}{L}$ and $\Gamma = \frac{2\mu}{E}$. This yields

$$\vec{\mathbf{M}} = \kappa_1 \vec{\mathbf{d}}_1 + \kappa_2 \vec{\mathbf{d}}_2 + \Gamma \kappa_3 \vec{\mathbf{d}}_3, \tag{5}$$

and

$$\vec{\mathbf{M}}' + \vec{\mathbf{d}}_3 \times \vec{\mathbf{F}} = \left(\vec{\mathbf{d}}_1 \times \ddot{\vec{\mathbf{d}}}_1 + \vec{\mathbf{d}}_2 \times \ddot{\vec{\mathbf{d}}}_2 \right), \tag{6}$$

where $\vec{\mathbf{M}}'$ is the torque. This yields a dimensionless balance of angular momentum which is equal to the balance of angular momentum employed by Goriely and Tabor [2].

V. LINEAR ANALYSIS WITH INERTIA AND WITHOUT DRAG

Section V is a reproduction of the work from Goriely and Tabor to facilitate the explanations of Section VI. Section VI is the linear analysis without inertia, with drag.

V.1. The Perturbation Expansion of the Basis Vectors

The vectors, $\vec{\mathbf{d}}_i^{(0)}$, are the basis vectors for our equilibrium solution, or the steady state (the straight, twisted rod under tension or compression). If the system starts at an equilibrium solution, it will stay there. We can write out the perturbation expansion for the local basis with the 0^{th} order terms (equilibrium) plus 1^{st} order term in ϵ (the small perturbation variable) plus higher order elements in ϵ:

$$\vec{\mathbf{d}}_i = \vec{\mathbf{d}}_i^{(0)} + \epsilon \vec{\mathbf{d}}_i^{(1)} + \mathcal{O}\left(\epsilon^2\right).$$

Henceforth, we ignore all terms $\mathcal{O}(\epsilon^2)$, which is consistent with previous work [1]. Thus, $\vec{\mathbf{d}}_i$ becomes

$$\vec{\mathbf{d}}_i = \vec{\mathbf{d}}_i^{(0)} + \epsilon \vec{\mathbf{d}}_i^{(1)}. \tag{7}$$

The equilibrium solution has an unperturbed basis, $\mathbf{d}_i^{(0)}$, so the 1^{st} order perturbation can be written as a rotation of the 0^{th} order basis

$$\vec{\mathbf{d}}_i^{(1)} = \mathbf{A}_{ij}\vec{\mathbf{d}}_j^{(0)},$$

where we again use the convention of summation over repeated indices. We require $\vec{\mathbf{d}}_i$ to be orthonormal at least to $\mathcal{O}(\epsilon)$, that is $\vec{\mathbf{d}}_i \cdot \vec{\mathbf{d}}_j = \delta_{ij}$. We exploit this to determine the elements of the matrix $\mathbf{A}$ where δ_{ij} is defined to be 1 when $i = j$ and 0 otherwise. Hence,

$$\left(\vec{\mathbf{d}}_i^{(0)} + \epsilon\vec{\mathbf{d}}_i^{(1)}\right) \cdot \left(\vec{\mathbf{d}}_j^{(0)} + \epsilon\vec{\mathbf{d}}_j^{(1)}\right) = \delta_{ij} + \epsilon\left(\mathbf{A}_{jk}\vec{\mathbf{d}}_i^{(0)} \cdot \vec{\mathbf{d}}_k^{(0)} + \mathbf{A}_{ik}\vec{\mathbf{d}}_j^{(0)} \cdot \vec{\mathbf{d}}_k^{(0)}\right). \quad (8)$$

Because we define our $\mathbf{d}_i^{(0)}$ to be orthonormal, by definition of dot products, $\mathbf{A}_{jk}\vec{\mathbf{d}}_i^{(0)} \cdot \vec{\mathbf{d}}_k^{(0)}$ only has a nonzero value when $k = i$ so replacing the k with i in this term results in

$$\left(\vec{\mathbf{d}}_i^{(0)} + \epsilon\vec{\mathbf{d}}_i^{(1)}\right) \cdot \left(\vec{\mathbf{d}}_j^{(0)} + \epsilon\vec{\mathbf{d}}_j^{(1)}\right) = \delta_{ij} + \epsilon\left(\mathbf{A}_{ji} + \mathbf{A}_{ik}\vec{\mathbf{d}}_j^{(0)} \cdot \vec{\mathbf{d}}_k^{(0)}\right).$$

Similarly, $\mathbf{A}_{ik}\vec{\mathbf{d}}_j^{(0)} \cdot \vec{\mathbf{d}}_k^{(0)}$ has nonzero value only when $k = j$, so plugging in k for j, this term is $\mathbf{A}_{ij}$. Thus, the $\mathcal{O}(\epsilon)$ term can be written as $\mathbf{A}_{ji} + \mathbf{A}_{ij}$. When substituting this back into Equation (8), we obtain

$$(\vec{\mathbf{d}}_i^{(0)} + \epsilon\vec{\mathbf{d}}_i^{(1)}) \cdot (\vec{\mathbf{d}}_j^{(0)} + \epsilon\vec{\mathbf{d}}_j^{(1)}) = \delta_{ij} + \epsilon(\mathbf{A}_{ji} + \mathbf{A}_{ij}).$$

Hence, we need $\mathbf{A}_{ji} + \mathbf{A}_{ij} = \vec{\mathbf{0}}$ since we require these vectors to be orthonormal to 1^{st} order. This means $\mathbf{A}$ can be written as an antisymmetric matrix

$$\mathbf{A} = \begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix},$$

leading to

$$\vec{\mathbf{d}}_i^{(1)} = \vec{\alpha} \times \vec{\mathbf{d}}_i^{(0)},$$

where $\vec{\alpha} = \alpha_i \vec{\mathbf{d}}_i^{(0)}$. Therefore we may rewrite $\vec{\mathbf{d}}_i$ as

$$\vec{\mathbf{d}}_i = \vec{\mathbf{d}}_i^{(0)} + \epsilon \left(\vec{\alpha} \times \vec{\mathbf{d}}_i^{(0)} \right).$$

The equation for $\vec{\mathbf{d}}_i$ is now in terms of what we know, $\vec{\mathbf{d}}_i^{(0)}$ and $\vec{\alpha}$. We have effectively written the three unknowns (each $\vec{\mathbf{d}}_i^{(1)}$) in terms of the three (α_i) by requiring the perturbed bases to remain orthonormal at least to first order. This means that we have reduced a total of nine unknowns to three.

V.2. Perturbation Expansion for an Arbitrary Vector

Let $\vec{\mathbf{V}}$ be an arbitrary vector. Then

$$\begin{aligned} \vec{\mathbf{V}} &= \vec{\mathbf{V}}^{(0)} + \epsilon \vec{\mathbf{V}}^{(1)} \\ &= v_i^{(0)} \vec{\mathbf{d}}_i^{(0)} + \epsilon \left(v_i^{(0)} \left(\vec{\alpha} \times \vec{\mathbf{d}}_i^{(0)} \right) + v_i^{(1)} \vec{\mathbf{d}}_i^{(0)} \right). \end{aligned}$$

The order ϵ component is $\vec{\mathbf{V}}^{(1)}$, so we can write

$$\begin{aligned} \vec{\mathbf{V}}^{(1)} = & \left(v_1^{(1)} + \alpha_2 v_3^{(0)} - \alpha_3 v_2^{(0)} \right) \vec{\mathbf{d}}_1^{(0)} + \left(v_2^{(1)} + \alpha_3 v_1^{(0)} - \alpha_1 v_3^{(0)} \right) \vec{\mathbf{d}}_2^{(0)} \\ & + \left(v_3^{(1)} + \alpha_1 v_2^{(0)} - \alpha_2 v_1^{(0)} \right) \vec{\mathbf{d}}_3^{(0)}. \end{aligned}$$

This is a way to expand an arbitrary vector and its derivative in terms of 0^{th} order components and 1^{st} components to use in our linear analysis. We will use this in calculating $\vec{\mathbf{F}}''$ in the balance of linear momentum.

V.3. The Perturbation for the Local Curvature and Twist

Recall $\vec{\kappa}$ and $\vec{\omega}$ are defined in Section II to be the bend and twist vectors in time and space respectively. The perturbation expansions for $\vec{\kappa}$ and $\vec{\omega}$ are

$$\vec{\kappa} = \vec{\kappa}^{(0)} + \epsilon \vec{\kappa}^{(1)}, \tag{9}$$

$$\vec{\omega} = \vec{\omega}^{(0)} + \epsilon \vec{\omega}^{(1)}. \tag{10}$$

Recall $\vec{\mathbf{d}}_i^{(1)} = \vec{\alpha} \times \vec{\mathbf{d}}_i^{(0)}$, and using Equation (1), the equation for $\left(\vec{\mathbf{d}}_i^{(1)}\right)'$ is

$$\left(\vec{\alpha} \times \vec{\mathbf{d}}_i^{(0)}\right)' = \vec{\kappa}^{(0)} \times \left(\vec{\alpha} \times \vec{\mathbf{d}}_i^{(0)}\right) + \vec{\kappa}^{(1)} \times \vec{\mathbf{d}}_i^{(0)}. \tag{11}$$

Taking the derivative with respect to space on the left and simplifying the right yield

$$\left(\vec{\alpha}' + \vec{\alpha} \times \vec{\kappa}_i^{(0)}\right) \times \vec{\mathbf{d}}_i^{(0)} = \vec{\kappa}^{(1)} \times \vec{\mathbf{d}}_i^{(0)}.$$

The triple product of arbitrary vectors $\vec{\mathbf{A}}, \vec{\mathbf{B}}, \vec{\mathbf{C}}$ is defined as

$$\left(\vec{\mathbf{A}} \times \vec{\mathbf{B}}\right) \times \vec{\mathbf{C}} = -\vec{\mathbf{A}}\left(\vec{\mathbf{B}} \cdot \vec{\mathbf{C}}\right) + \vec{\mathbf{B}}\left(\vec{\mathbf{A}} \cdot \vec{\mathbf{C}}\right).$$

After application of this identity, we obtain

$$\vec{\kappa}^{(1)} = \vec{\alpha}' + \vec{\kappa}^{(0)} \times \vec{\alpha}. \tag{12}$$

Since $\vec{\omega}^{(0)} = \vec{\mathbf{0}}$ because the stationary solution is constant in time and $\dot{\vec{\mathbf{d}}}_i = \vec{\omega} \times \vec{\mathbf{d}}_i$, $\vec{\omega}^{(1)} = \dot{\vec{\alpha}}$. We now have reduced our problem from nine unknowns ($\vec{\mathbf{d}}_i^{(1)}, \vec{\kappa}^{(1)}$, and $\vec{\omega}^{(0)}$) to three unknowns (α_i). Using Sections V.1 and V.3 we have reduced fifteen unknowns to three.

V.4. Balance of Linear Momentum

Recall from Section IV.1, the balance of linear momentum is given by Equation (2). For simplicity, we will address each side of this equation separately.

V.4.1. Left-Hand Side of the Balance of Linear Momentum

The left-hand side of the balance of linear momentum can be written as

$$\vec{\mathbf{F}}'' = \left(\vec{\mathbf{F}}^{(0)}\right)'' + \epsilon\left(\vec{\mathbf{F}}^{(1)}\right)''.$$

Using Section V.2, the force can be expanded

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}^{(0)} + \epsilon\vec{\mathbf{F}}^{(1)} = f_i^{(0)}\vec{\mathbf{d}}_i^{(0)} + \epsilon\Big(f_i^{(1)} + \Big(\vec{\alpha} \times f_j^{(0)}\vec{\mathbf{d}}_j^{(0)}\Big)_i\Big)\vec{\mathbf{d}}_i^{(0)}.$$

From this we can write

$$\vec{\mathbf{F}}^{(1)} = \left(f_i^{(1)} + \left(\vec{\alpha} \times f_j^{(0)}\vec{\mathbf{d}}_j^{(0)}\right)_i\right)\vec{\mathbf{d}}_i^{(0)}. \tag{13}$$

For an arbitrary vector $\vec{\mathbf{F}}$,

$$\vec{\mathbf{F}} = \vec{\mathbf{F}}^{(0)} + \epsilon\vec{\mathbf{F}}^{(1)}.$$

This can be used to calculate the derivatives with respect to space and time of an arbitrary vector to solve for our unknowns. Let $\vec{\mathbf{F}}^{(1)} = v_i\vec{\mathbf{d}}_i^{(0)}$, so

$$v_i = f_i^{(1)} + \left(\vec{\alpha} \times f_j^{(0)}\vec{\mathbf{d}}_j^{(0)}\right)_i.$$

The cross product of $\vec{\alpha}$ and $\vec{\mathbf{f}}^{(0)}$ yields

$$\vec{\alpha} \times \vec{\mathbf{f}}^{(0)} = \left(\alpha_2 f_3^{(0)} - \alpha_3 f_2^{(0)}\right)\vec{\mathbf{d}}_1^{(0)} - \left(\alpha_1 f_3^{(0)} - \alpha_3 f_1^{(0)}\right)\vec{\mathbf{d}}_2^{(0)} + \left(\alpha_1 f_2^{(0)} - \alpha_2 f_1^{(0)}\right)\vec{\mathbf{d}}_3^{(0)}.$$

From this we can read off the components,

$$\begin{aligned} v_1 &= f_1^{(1)} + \alpha_2 f_3^{(0)} - \alpha_3 f_2^{(0)}, \\ v_2 &= f_2^{(1)} + \alpha_3 f_1^{(0)} - \alpha_1 f_3^{(0)}, \\ v_3 &= f_3^{(1)} + \alpha_1 f_2^{(0)} - \alpha_2 f_1^{(0)}. \end{aligned}$$

The derivative is given by

$$\vec{\mathbf{F}}' = v_i'\vec{\mathbf{d}}_i + v_i\left(\vec{\kappa} \times \vec{\mathbf{d}}_i\right). \tag{14}$$

The summation can be written out as

$$\vec{\mathbf{F}}' = \left(v_1' - v_2\kappa_3^{(0)} + v_3\kappa_2^{(0)}\right)\vec{\mathbf{d}}_1 + \left(v_2' + v_1\kappa_3^{(0)} - v_3\kappa_1^{(0)}\right)\vec{\mathbf{d}}_2 + \left(v_3' - v_1\kappa_2^{(0)} + v_2\kappa_1^{(0)}\right)\vec{\mathbf{d}}_3.$$

The second derivative of $\vec{\mathbf{F}}$ is

$$\begin{aligned} \vec{\mathbf{F}}'' &= v_i''\vec{\mathbf{d}}_i^{(0)} + v_i'\vec{\mathbf{d}}_i^{(0)\prime} + v_i'\left(\vec{\kappa} \times \vec{\mathbf{d}}_i^{(0)}\right) + v_i\vec{\kappa}' \times \vec{\mathbf{d}}_i^{(0)} + v_i\vec{\kappa} \times \vec{\mathbf{d}}_i^{(0)\prime} \\ &= \left(v_i'' - 2\left(v_j'\vec{\mathbf{d}}_j^{(0)} \times \vec{\kappa}\right)_i - \left(v_j\vec{\mathbf{d}}_j^{(0)} \times \vec{\kappa}'\right)_i + \left(v_j\vec{\mathbf{d}}_j^{(0)} \cdot \vec{\kappa}\right)\vec{\kappa}_i - \vec{\kappa}^2 v_i\right)\vec{\mathbf{d}}_i^{(0)}. \end{aligned}$$

V.4.2. Right-Hand Side of the Balance of Linear Momentum

The $O(\epsilon)$ component of the balance of linear momentum is

$$\ddot{\vec{\mathbf{d}}}_3^{(1)} = \ddot{\alpha}_2 \vec{\mathbf{d}}_1^{(0)} - \ddot{\alpha}_1 \vec{\mathbf{d}}_2^{(0)},$$

because the $\vec{\mathbf{d}}_i^{(0)}$ vectors do not depend on time.

V.4.3. Full Balance of Linear Momentum

Matching up the left and right side of the balance of linear momentum we have

$$\begin{aligned}
\ddot{\alpha}_2 &= \left(f_1^{(1)}\right)'' + \alpha_2'' f_3^{(0)} - 2\kappa_3^{(0)} \left(\left(f_2^{(1)}\right)' - \alpha_1' f_3^{(0)} \right) - \left(\kappa_3^{(0)}\right)^2 \left(f_1^{(1)} + \alpha_2 f_3^{(0)}\right), \\
-\ddot{\alpha}_1 &= \left(f_2^{(1)}\right)'' - \alpha_1'' f_3^{(0)} + 2\kappa_3^{(0)} \left(\left(f_1^{(1)}\right)' + \alpha_2' f_3^{(0)} \right) - \left(\kappa_3^{(0)}\right)^2 \left(f_2^{(1)} - \alpha_1 f_3^{(0)}\right), \\
0 &= \left(f_3^{(1)}\right)''.
\end{aligned}$$

This balance yields three equations and six unknowns $\left(f_1^{(1)}, f_2^{(1)}, f_3^{(1)}, \alpha_1, \alpha_2, \alpha_3\right)$. The remaining three equations will come from the balance of angular momentum in Section V.5.

V.5. Balance of Angular Momentum

Like in the balance of linear momentum, here we will handle the left-hand and right-hand sides of the balance of angular momentum separately.

V.5.1. Left-Hand Side of the Balance of Angular Momentum

From Equation (6) the left-hand side of the balance of angular momentum is $\vec{\mathbf{M}}' + \vec{\mathbf{d}}_3 \times \vec{\mathbf{F}}$. The relationship between the resultant torque, $\vec{\mathbf{M}}$, and the bend and twist of the rod, κ, is given by Equation (5). Substituting κ_i and $\vec{\mathbf{d}}_i$ with their perturbed expansions Equations (9) and (10),

$$\vec{\mathbf{M}} = \left(\kappa_\beta^{(0)} + \epsilon\kappa_\beta^{(1)}\right)\left(\vec{\mathbf{d}}_\beta^{(0)} + \epsilon\vec{\mathbf{d}}_\beta^{(1)}\right) + \Gamma\left(\kappa_3^{(0)} + \epsilon\kappa_3^{(1)}\right)\left(\vec{\mathbf{d}}_3^{(0)} + \epsilon\vec{\mathbf{d}}_3^{(1)}\right).$$

Distributing the terms and looking at the 1^{st} order component of each side,

$$\begin{aligned}\vec{\mathbf{M}}^{(1)} = &\left(\kappa_1^{(1)} - \kappa_2^{(0)}\alpha_3 + \Gamma\kappa_3^{(0)}\alpha_2\right)\vec{\mathbf{d}}_1^{(0)} + \left(\kappa_2^{(1)} - \Gamma\kappa_3^{(0)}\alpha_1 + \kappa_1^{(0)}\alpha_3\right)\vec{\mathbf{d}}_2^{(0)} \\ &+ \left(\Gamma\kappa_3^{(1)} - \kappa_1^{(0)}\alpha_2 + \kappa_2^{(0)}\alpha_1\right)\vec{\mathbf{d}}_3^{(0)}.\end{aligned} \tag{15}$$

Writing out $\vec{\alpha}' + \vec{\kappa}^{(0)} \times \vec{\alpha}$ and substituting it into the right-hand side of Equation (12) yields

$$\begin{aligned}\kappa^{(1)} = &\left(\alpha_1' + \kappa_2^{(0)}\alpha_3 - \kappa_3^{(0)}\alpha_2\right)\vec{\mathbf{d}}_1^{(0)} + \left(\alpha_2' - \kappa_1^{(0)}\alpha_3 + \kappa_3^{(0)}\alpha_1\right)\vec{\mathbf{d}}_2^{(0)} \\ &+ \left(\alpha_3' + \kappa_1^{(0)}\alpha_2 - \kappa_2^{(0)}\alpha_1\right)\vec{\mathbf{d}}_3^{(0)},\end{aligned}$$

where we define $\kappa_i^{(1)} = \vec{\kappa}^{(1)} \cdot \vec{\mathbf{d}}_i^{(0)}$. We can substitute those values into Equation (15), yielding

$$\begin{aligned}\vec{\mathbf{M}}^{(1)} = &\left(\alpha_1' + (\Gamma - 1)\,\kappa_3^{(0)}\alpha_2\right)\vec{\mathbf{d}}_1^{(0)} + \left(\alpha_2' + (1 - \Gamma)\,\kappa_3^{(0)}\alpha_1\right)\vec{\mathbf{d}}_2^{(0)} \\ &+ \left(\Gamma\alpha_3' + (\Gamma - 1)\,\kappa_1^{(0)}\alpha_2 + (1 - \Gamma)\,\kappa_2^{(0)}\alpha_1\right)\vec{\mathbf{d}}_3^{(0)}.\end{aligned}$$

We can use Equation (14) and compute the derivatives to obtain

$$\begin{aligned}\vec{\mathbf{M}}^{(1)\prime} = &\left(\alpha_1'' + (\Gamma - 1)\left(\kappa_3^{(0)\prime}\alpha_2 + \kappa_3^{(0)}\alpha_2'\right) - \kappa_3^{(0)}\alpha_2' - \kappa_3^{(0)}\,(1 - \Gamma)\,\kappa_3^{(0)}\alpha_1\right)\vec{\mathbf{d}}_1^{(0)} \\ &+ \left(\kappa_2^{(0)}\Gamma\alpha_3' + \kappa_2^{(0)}\,(\Gamma - 1)\,\kappa_1^{(0)}\alpha_2 + \kappa_2^{(0)}\,(1 - \Gamma)\,\kappa_2^{(0)}\alpha_1\right)\vec{\mathbf{d}}_1^{(0)} \\ &+ \left(\alpha_2'' + (1 - \Gamma)\left(\kappa_3^{(0)\prime}\alpha_1 + \kappa_3^{(0)}\alpha_1'\right)\right)\vec{\mathbf{d}}_2^{(0)} \\ &+ \left(\kappa_3^{(0)}\alpha_1' + \kappa_3^{(0)}\,(\Gamma - 1)\,\kappa_3^{(0)}\alpha_2\right)\vec{\mathbf{d}}_2^{(0)} \\ &+ \left(-\kappa_1^{(0)}\Gamma\alpha_3' - \kappa_1^{(0)}\,(\Gamma - 1)\,\kappa_1^{(0)}\alpha_2 - \kappa_1^{(0)}\,(1 - \Gamma)\,\kappa_2^{(0)}\alpha_1\right)\vec{\mathbf{d}}_2^{(0)} \\ &+ \left(\Gamma\alpha_3'' + (\Gamma - 1)\left(\kappa_1^{(0)\prime}\alpha_2 + \kappa_1^{(0)}\alpha_2'\right) + (1 - \Gamma)\left(\kappa_2^{(0)\prime}\alpha_1 + \kappa_2^{(0)}\alpha_1'\right)\right)\vec{\mathbf{d}}_3^{(0)} \\ &+ \left(-\kappa_2^{(0)}\alpha_1' - \kappa_2^{(0)}\,(\Gamma - 1)\,\kappa_3^{(0)}\alpha_2 + \kappa_1^{(0)}\alpha_2' + \kappa_1^{(0)}\,(1 - \Gamma)\,\kappa_3^{(0)}\alpha_1\right)\vec{\mathbf{d}}_3^{(0)}.\end{aligned} \tag{16}$$

For the component that represents the torque that results from the resultant force in the balance of angular momentum, we need to calculate $\vec{\mathbf{d}}_3 \times \vec{\mathbf{F}}$,

$$\vec{\mathbf{d}}_3 \times \vec{\mathbf{F}} = \vec{\mathbf{d}}_3^{(0)} \times \vec{\mathbf{F}}^{(0)} + \epsilon \left(\vec{\mathbf{d}}_3^{(0)} \times \vec{\mathbf{F}}^{(1)} + \alpha_2 \vec{\mathbf{d}}_1^{(0)} \times \vec{\mathbf{F}}^{(0)} - \alpha_1 \vec{\mathbf{d}}_2^{(0)} \times \vec{\mathbf{F}}^{(0)} \right).$$

The first order component (or everything multiplied by ϵ) is the only part that will yield our unknowns, as

$$\begin{aligned} \left(\vec{\mathbf{d}}_3 \times \vec{\mathbf{F}} \right)^{(1)} &= \left(-f_2^{(1)} - \alpha_3 f_1^{(0)} \right) \vec{\mathbf{d}}_1^{(0)} + \left(f_1^{(1)} - \alpha_3 f_2^{(0)} \right) \vec{\mathbf{d}}_2^{(0)} \\ &+ \left(\alpha_2 f_2^{(0)} + \alpha_1 f_1^{(0)} \right) \vec{\mathbf{d}}_3^{(0)}. \end{aligned} \tag{17}$$

Adding Equations (16) and (17) will give us the left-hand side of the balance of angular momentum.

V.5.2. Right-Hand Side of the Balance of Angular Momentum

The right-hand side of the equation is represented by $\vec{\mathbf{d}}_1 \times \ddot{\vec{\mathbf{d}}}_1 + \vec{\mathbf{d}}_2 \times \ddot{\vec{\mathbf{d}}}_2$. Because $\vec{\mathbf{d}}_1^{(0)}$ is constant in time, $\dot{\vec{\mathbf{d}}}_1^{(0)} = \vec{\mathbf{0}}$. Recall $\ddot{\vec{\mathbf{d}}}_1 = \epsilon \left(\ddot{\vec{\alpha}} \times \vec{\mathbf{d}}_1^{(0)} \right)$, and we know $\ddot{\vec{\alpha}} = \ddot{\alpha}_i \vec{\mathbf{d}}_i^{(0)}$, hence the unknown parts of $\ddot{\vec{\mathbf{d}}}_1$, or the ϵ component, are $-\ddot{\alpha}_2 \vec{\mathbf{d}}_3^{(0)} + \ddot{\alpha}_3 \vec{\mathbf{d}}_2^{(0)}$, so $\vec{\mathbf{d}}_1 \times \ddot{\vec{\mathbf{d}}}_1 = \epsilon \left(\ddot{\alpha}_2 \vec{\mathbf{d}}_2^{(0)} + \ddot{\alpha}_3 \vec{\mathbf{d}}_3^{(0)} \right)$. Similarly, the unknown parts of $\vec{\mathbf{d}}_2 \times \ddot{\vec{\mathbf{d}}}_2$ are $\ddot{\alpha}_1 \vec{\mathbf{d}}_1^{(0)} + \ddot{\alpha}_3 \vec{\mathbf{d}}_3^{(0)}$. Adding the 1st order components of the two cross products results in

$$\vec{\mathbf{d}}_1 \times \ddot{\vec{\mathbf{d}}}_1 + \vec{\mathbf{d}}_2 \times \ddot{\vec{\mathbf{d}}}_2 = \epsilon \left(\ddot{\alpha}_1 \vec{\mathbf{d}}_1^{(0)} + \ddot{\alpha}_2 \vec{\mathbf{d}}_2^{(0)} + 2 \ddot{\alpha}_3 \vec{\mathbf{d}}_3^{(0)} \right).$$

The right-hand side of the balance of angular momentum is our ϵ component, or $\ddot{\alpha}_1 \vec{\mathbf{d}}_1^{(0)} + \ddot{\alpha}_2 \vec{\mathbf{d}}_2^{(0)} + 2 \ddot{\alpha}_3 \vec{\mathbf{d}}_3^{(0)}$.

V.5.3. Full Balance of Angular Momentum

Substituting Equations (16) and (17) into Equation (6), the balance of angular momentum gives three first order equations in ϵ as follows:

$$\begin{aligned}
\ddot{\alpha}_1 &= \alpha_1'' + (\Gamma-1)\left(\kappa_3^{(0)\prime}\alpha_2\right) + (\Gamma-2)\,\kappa_3^{(0)}\alpha_2' - (1-\Gamma)\left(\kappa_3^{(0)}\right)^2\alpha_1 \\
&\quad +\kappa_2^{(0)}\Gamma\alpha_3' + \kappa_2^{(0)}(\Gamma-1)\,\kappa_1^{(0)}\alpha_2 + (1-\Gamma)\left(\kappa_2^{(0)}\right)^2\alpha_1 - f_2^{(1)} - \alpha_3 f_1^{(0)}, \\
\ddot{\alpha}_2 &= \alpha_2'' + (1-\Gamma)\left(\kappa_3^{(0)\prime}\alpha_1\right) + (2-\Gamma)\,\kappa_3^{(0)}\alpha_1' + \kappa_3^{(0)}(\Gamma-1)\,\kappa_3^{(0)}\alpha_2 \\
&\quad -\kappa_1^{(0)}\Gamma\alpha_3' - \kappa_1^{(0)}(\Gamma-1)\,\kappa_1^{(0)}\alpha_2 - \kappa_1^{(0)}(1-\Gamma)\,\kappa_2^{(0)}\alpha_1 + f_1^{(1)} - \alpha_3 f_2^{(0)}, \\
2\ddot{\alpha}_3 &= \Gamma\alpha_3'' + (\Gamma-1)\left(\kappa_1^{(0)\prime}\alpha_2 + \kappa_1^{(0)}\alpha_2'\right) + (1-\Gamma)\left(\kappa_2^{(0)\prime}\alpha_1 + \kappa_2^{(0)}\alpha_1'\right) \\
&\quad -\kappa_2^{(0)}\alpha_1' - \kappa_2^{(0)}(\Gamma-1)\,\kappa_3^{(0)}\alpha_2 + \kappa_1^{(0)}\alpha_2' + \kappa_1^{(0)}(1-\Gamma)\,\kappa_3^{(0)}\alpha_1 + \alpha_2 f_2^{(0)} + \alpha_1 f_1^{(0)}.
\end{aligned}$$

These represent the remaining three equations with six unknowns.

V.6. Six Equations, Six Unknowns

Writing out the six equations that are linear in our six unknowns and substituting in $f_1^{(0)} = f_2^{(0)} = \kappa_1^{(0)} = \kappa_2^{(0)} = 0$ since the rod is straight yields

$$\begin{aligned}
\ddot{\alpha}_2 &= \left(f_1^{(1)}\right)'' + \alpha_2'' f_3^{(0)} - 2\kappa_3^{(0)}\left(\left(f_2^{(1)}\right)' - \alpha_1' f_3^{(0)}\right) - \left(\kappa_3^{(0)}\right)^2\left(f_1^{(1)} + \alpha_2 f_3^{(0)}\right), \\
-\ddot{\alpha}_1 &= \left(f_2^{(1)}\right)'' - \alpha_1'' f_3^{(0)} + 2\kappa_3^{(0)}\left(\left(f_1^{(1)}\right)' + \alpha_2' f_3^{(0)}\right) - \left(\kappa_3^{(0)}\right)^2\left(f_2^{(1)} - \alpha_1 f_3^{(0)}\right), \\
0 &= \left(f_3^{(1)}\right)'', \\
\ddot{\alpha}_1 &= \alpha_1'' + (\Gamma-2)\,\kappa_3^{(0)}\alpha_2' - (1-\Gamma)\left(\kappa_3^{(0)}\right)^2\alpha_1 - f_2^{(1)}, \\
\ddot{\alpha}_2 &= \alpha_2'' + (2-\Gamma)\,\kappa_3^{(0)}\alpha_1' + \kappa_3^{(0)}(\Gamma-1)\,\kappa_3^{(0)}\alpha_2 + f_1^{(1)}, \\
2\ddot{\alpha}_3 &= \Gamma\alpha_3''.
\end{aligned}$$

For $j \in \{1,2,3\}$, we assume the standard form of the perturbation to be

$$\alpha_j = e^{\sigma t}\left(\hat{\alpha}_j e^{ins} + \tilde{\hat{\alpha}}_j e^{-ins}\right), \tag{18}$$

$$f_j^{(1)} = e^{\sigma t}\left(\hat{f}_j e^{ins} + \tilde{\hat{f}}_j e^{-ins}\right), \tag{19}$$

where $\tilde{()}$ represents the complex conjugate. The parameter σ will determine the stability of our steady state. When $\mathrm{Re}(\sigma)$ is positive, the perturbations grow and our rod is unstable and when $\mathrm{Re}(\sigma)$ is negative, the perturbations decay and our steady state is stable. When $\sigma = 0$, we have neutral modes where our perturbations neither grow nor decay in time. From previous work, we know $f_3^{(0)} = P^2$ and $\kappa_3^{(0)} = \gamma$ [2]. Following the general procedure for determining the eigenvalues of a linearized PDE system, the matrix equation that corresponds to our case with inertia, without drag is

$$\mathbf{A} = \begin{pmatrix} 2\gamma i n P^2 & -\gamma^2 P^2 - \sigma^2 - n^2 P^2 & 0 & -n^2 - \gamma^2 & -2\gamma i n & 0 \\ \sigma^2 + n^2 P^2 + \gamma^2 P^2 & 2\gamma i n P^2 & 0 & 2\gamma i n & -n^2 - \gamma^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -n^2 \\ -\sigma^2 - n^2 - (1-\Gamma)\gamma^2 & (\Gamma - 2)\gamma i n & 0 & 0 & -1 & 0 \\ (2-\Gamma)\gamma i n & (\Gamma - 1)\gamma^2 - \sigma^2 - n^2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2\sigma^2 - \Gamma n^2 & 0 & 0 & 0 \end{pmatrix}.$$

To determine stability, we compute $\det(\mathbf{A})$ and set it equal to zero. We want to know the stability of the steady state. To reiterate, the steady state is stable when $\mathrm{Re}(\sigma)$ is negative and stable when $\mathrm{Re}(\sigma)$ is positive. Hence, we are looking for a change in sign of σ, so we first look at our neutral modes when $\sigma = 0$. With substitution of $\bar{X} = (\gamma^2 - n^2)^2 + \gamma^2 + n^2$, $\bar{Y} = P^2 + n^2 + \gamma^2(1-\Gamma)$, and considering when $\det \mathbf{A} = 0$, we find the following quadratic equation.

$$\left(\bar{X} + n^2 + \gamma^2 + 1\right)\sigma^4 + \left(2\bar{X}\bar{Y} + 4\gamma^2 n^2 (2-\Gamma)\right)\sigma^2 + \left(n^2 - \gamma^2\right)^2\left(\bar{Y}^2 - \gamma^2 n^2 (2-\Gamma)^2\right) = 0. \tag{20}$$

But this is quadratic in σ^2 such that

$$\sigma^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

where

$$\begin{aligned}
A &= \bar{X} + n^2 + \gamma^2 + 1, \\
B &= 2\left(\bar{X}\bar{Y} + 2\gamma^2 n^2 (2-\Gamma)\right), \\
C &= \left(\bar{X} - \gamma^2 - n^2\right)\left(\bar{Y}^2 - \gamma^2 n^2 (2-\Gamma)^2\right).
\end{aligned}$$

Noting that $B^2 - 4AC$ is a perfect square, we can write

$$\sigma^2 = \frac{-\bar{X}\bar{Y} - 2\gamma^2 n^2 (2-\Gamma) \pm \gamma n \left((2-\Gamma)\bar{X} + 2\bar{Y}\right)}{\left(\bar{X} + n^2 + \gamma^2 + 1\right)}. \tag{21}$$

We let the larger value of σ^2 in Equation (21) be σ_1^2 and the lesser be σ_2^2. By taking the square root of both sides of this equation, we can calculate σ. Because σ^2 is equal to a real number, when $\sigma^2 > 0$, we have instability because σ will be plus or minus the square root of a real number. If $\sigma^2 < 0$, σ will be purely imaginary, meaning we will have oscillations in time that neither grow nor decay.

V.6.1. Neutral Modes

To determine the neutral modes, we solve Equation (20) when $\sigma = 0$. This yields $0 = \left(n^2 - \gamma^2\right)^2 \left(\bar{Y}^2 - \gamma^2 n^2 (2-\Gamma)^2\right)$, which has two solutions. Either $n^2 = \gamma^2$ or $\bar{Y}^2 = \gamma^2 n^2 (2-\Gamma)^2$. Recall $\bar{Y} = P^2 + n^2 + \gamma^2 (1-\Gamma)$, hence

$$n^2 - \gamma (2-\Gamma) n + P^2 + \gamma^2 (1-\Gamma) = 0,$$

which is a quadratic in n so

$$n = \frac{\gamma (2-\Gamma) \pm \sqrt{\gamma^2 (2-\Gamma)^2 - 4\left(P^2 + \gamma^2 (1-\Gamma)\right)}}{2}.$$

The radicand changes from positive (so the roots are real) to negative (the roots have a complex value) across $\gamma_c = \frac{2P}{\Gamma}$. We call γ_c the critical value of γ. On this curve (the red curve in Figure 2), we can solve for the critical value of n.

$$n_c = \frac{P\,(2-\Gamma)}{\Gamma}.$$

To illustrate this, Figure 2 shows the $n = \gamma$ curve (blue) and the hyperbola (red) for the neutral modes when $\sigma = 0$ for $\Gamma = \frac{2}{3}$ and $P = 3$.

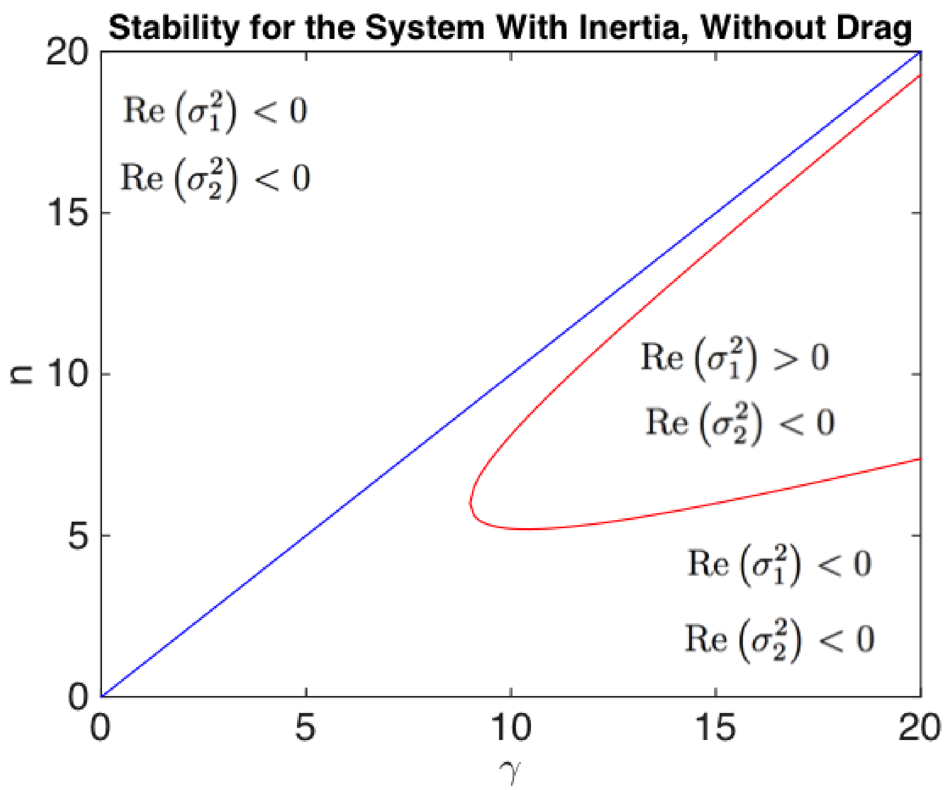

FIG. 2. The blue and red curves represent the neutral modes ($\sigma = 0$) when $n = \gamma$ (blue) and the hyperbola (red) for $\Gamma = \frac{2}{3}$ and $P = 3$.

The two curves in Figure 2 are the neutral modes when $\sigma = 0$, meaning that on these curves the perturbation solution of the rod neither grows nor decays in time. Since σ_i^2 is always a real number, when it is positive, we will have one value of σ a positive real number and one value a negative real number. The fact that one value is positive indicates that our perturbations will grow in time. That is, our steady state is unstable. When σ^2 is negative, both values of σ will be complex, indicating that we expect to see our perturbation as oscillations, where these oscillations neither grow nor decay in time. Essentially, the main result of this linear analysis with inertia, without drag is Figure 2. This figure shows that the straight rod with twist under tension or compression is unstable after reaching a critical twist, which matches with intuition. At this critical value of γ, as long as the wave number lies on the red curve, then a rod buckles into an unstable helix.

VI. LINEAR ANALYSIS WITHOUT INERTIA AND WITH DRAG

Up until this point, we have been reproducing the work of Goriely and Tabor [2]. This has been done to lay down the groundwork for the model we will be studying. From here, we alter the model to include hydrodynamic drag while removing inertia. This subsequent analysis is achieved by first altering our equations for the balance of torque and force. From there we can conduct the analysis for this new case.

VI.1. Balance of Linear Momentum

We begin by altering the balance of linear momentum, Equation (2), to incorporate drag and to remove inertia. Hydrodynamic drag can be reasonably approximated with resistive force theory, which says that the resistance the fluid provides to the rod is directly proportional to the velocity of the centerline of the rod [4]. Resistive force theory says that the fluid will resist the movement of the rod somewhat in proportion to $\dot{\vec{\mathbf{r}}}_0$, which is twice the size in the $\vec{\mathbf{d}}_1$ and $\vec{\mathbf{d}}_2$ direction as it is in the $\vec{\mathbf{d}}_3$ direction. After matrix algebra manipulations and nondimensionalization, this yields Equation (22).

$$\left[\left(\vec{\mathbf{F}}' \cdot \vec{\mathbf{d}}_1\right) \vec{\mathbf{d}}_1 + \left(\vec{\mathbf{F}}' \cdot \vec{\mathbf{d}}_2\right) \vec{\mathbf{d}}_2 + 2\left(\vec{\mathbf{F}}' \cdot \vec{\mathbf{d}}_3\right) \vec{\mathbf{d}}_3\right]' = \dot{\vec{\mathbf{d}}}_3. \tag{22}$$

Using the same techniques as developed in Section V.5.1 and simplifying, the first order components of Equation (22) are now

$$\begin{aligned}
0 &= -\dot{\alpha}_2 + \left(f_1^{(1)}\right)'' + \alpha_2'' P^2 - 2\gamma \left(f_2^{(1)}\right)' + 2\gamma\alpha_1' P^2 - \gamma^2 f_1^{(1)} - \gamma^2 \alpha_2 P^2, \\
0 &= \dot{\alpha}_1 + \left(f_2^{(1)}\right)'' - \alpha_1'' P^2 + 2\gamma \left(f_1^{(1)}\right)' + 2\gamma\alpha_2' P^2 - \gamma^2 f_2^{(1)} + \gamma^2 \alpha_1 P^2, \\
0 &= 2\left(f_3^{(1)}\right)''.
\end{aligned}$$

These represent three of our required six equations. We will obtain the remaining equations with the balance of angular momentum in Section VI.2.

VI.2. Balance of Angular Momentum

We alter our previous balance of angular momentum, Equation (6), from Section IV.2 by removing inertia, as seen in Equation (23).

$$\vec{\mathbf{M}}' + \vec{\mathbf{d}}_3 \times \vec{\mathbf{F}} = 0. \tag{23}$$

No additional component is added to represent drag because any torque due to drag would have to be multiplied by the lever arm on which it acts. In this model, the length of the lever arm is the radius, which is small. Hence we would only be adding terms of size $\left(\frac{a}{L}\right)^2$, which by design are excluded by Kirchhoff theory. Using the same techniques as developed in Section V.5.1 and simplifying, the first order components of Equation 23 are now

$$
\begin{aligned}
0 &= \alpha_1'' + (\Gamma - 2)\,\gamma\alpha_2' - (1-\Gamma)\,(\gamma)^2\,\alpha_1 - f_2^{(1)},\\
0 &= \alpha_2'' + (2-\Gamma)\,\gamma\alpha_1' + \gamma\,(\Gamma-1)\,\gamma\alpha_2 + f_1^{(1)},\\
0 &= \Gamma\alpha_3''.
\end{aligned}
$$

These represent the last three of our required six equations. We obtained the other equations with the balance of linear momentum in Section VI.1.

VI.3. Balance of Linear and Angular Momentum with Drag

Combining the new results of Sections VI.1 and VI.2, we have six equations with six unknowns:

$$
\begin{aligned}
0 &= -\dot{\alpha}_2 + \left(f_1^{(1)}\right)'' + \alpha_2''P^2 - 2\gamma\left(f_2^{(1)}\right)' + 2\gamma\alpha_1'P^2 - \gamma^2 f_1^{(1)} - \gamma^2\alpha_2 P^2\\
0 &= \dot{\alpha}_1 + \left(f_2^{(1)}\right)'' - \alpha_1''P^2 + 2\gamma\left(f_1^{(1)}\right)' + 2\gamma\alpha_2'P^2 - \gamma^2 f_2^{(1)} + \gamma^2\alpha_1 P^2\\
0 &= 2\left(f_3^{(1)}\right)''\\
0 &= \alpha_1'' + (\Gamma - 2)\,\gamma\alpha_2' - (1-\Gamma)\,(\gamma)^2\,\alpha_1 - f_2^{(1)},\\
0 &= \alpha_2'' + (2-\Gamma)\,\gamma\alpha_1' + \gamma\,(\Gamma-1)\,\gamma\alpha_2 + f_1^{(1)},\\
0 &= \Gamma\alpha_3''.
\end{aligned}
$$

Using procedures from Section V.6, we obtain the new matrix $\mathbf{A}$ as

$$
\mathbf{A} = \begin{pmatrix}
2\gamma i n P^2 & -\sigma - (n^2+\gamma^2)\,P^2 & 0 & -(n^2+\gamma^2) & -2\gamma i n & 0\\
\sigma + (n^2+\gamma^2)\,P^2 & 2\gamma i n P^2 & 0 & 2\gamma i n & -(n^2+\gamma^2) & 0\\
0 & 0 & 0 & 0 & 0 & -2n^2\\
-n^2 - (1-\Gamma)\,\gamma^2 & -(2-\Gamma)\,\gamma i n & 0 & 0 & -1 & 0\\
(2-\Gamma)\,\gamma i n & -n^2 - (1-\Gamma)\,\gamma^2 & 0 & 1 & 0 & 0\\
0 & 0 & -\Gamma n^2 & 0 & 0 & 0
\end{pmatrix}.
$$

Taking the determinant of this matrix and setting it equal to zero,

$$\sigma^2 + 2\left[\left(n^2+\gamma^2\right)\bar{Y} + 2\gamma^2 n^2 \left(2-\Gamma\right)\right]\sigma + \left(n^2-\gamma^2\right)^2\left(\bar{Y}^2 - \gamma^2 n^2 \left(2-\Gamma\right)^2\right) = 0. \quad (24)$$

Once again, when $\sigma = 0$, we have the neutral modes as discussed in Section V.6.1. The neutral modes are identical to those for the case with inertia, without drag. This makes qualitative sense because the only changes in our equations were in the time-dependent terms. We also let the larger value of σ in Equation (24) be σ_1 and the lesser be σ_2.

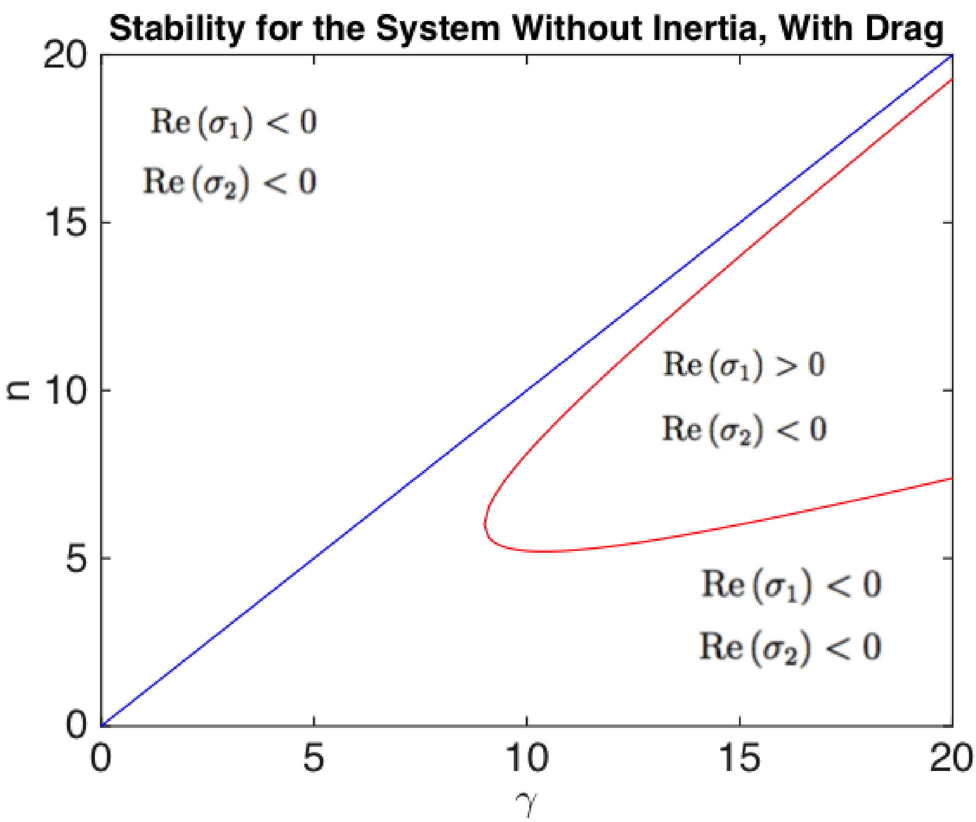

FIG. 3. The blue and red curves represent the neutral modes ($\sigma = 0$) when $n = \gamma$ (blue) and the hyperbola (red) for $\Gamma = \frac{2}{3}$ and $P = 3$.

Stability again comes down to whether or not $\text{Re}(\sigma)$ is positive in the regions bisected by the curves. Similar to the previous case, $\text{Re}(\sigma_1)$ is positive inside the red curve, indicating that the perturbations will grow in time. The rod perturbed into the region where the real parts of both σs are negative returns to steady state. Once again, the straight twisted rod held under tension or compression is stable until a critical value of twist, γ_c, after which, the rod buckles into an unstable helix. The primary result of this work is Figure 3. These results are somewhat surprising because intuitively, we would expect a rod with external damping to always return to its steady state, or, if it does not, this critical value should be dependent on the size of the external damping. However, this figure illustrates that this is not the case.

VII. CONCLUSION

In this work, we study the linear analysis of the Kirchhoff rod both with inertia, lacking drag and with drag, lacking inertia. In both cases, the rods were assumed to be linearly elastic. Interestingly, both models have the same steady state results and very similar stability results (Figure 2 and Figure 3). When solving for the real parts of σ^2 and σ, we see the same stability regions have apparent similar signs, which is unexpected. In the case with inertia, our results tell us that either the rod grows with respect to time or we observe oscillations for infinite time. On the other hand, in the case with drag, although we again observe that the perturbations may grow with respect to time, we also see that, under the right conditions, the rod will return to its steady state and is therefore stable. It is an unexpected result to see any positive $\text{Re}(\sigma)$ values for the case with drag, without inertia. This is because we would expect hydrodynamic drag to have a negative effect on any perturbations. One reason for this could be that the drag does not affect the dynamics of the slender rod as much as other variables, like its internal forces.

For future work, we plan to add in a viscoelastic constitutive relation to make the rod more worm-like. We expect living organisms to act in a more viscoelastic way than purely elastic because of the response of muscle dynamics. We intend to repeat this process for the Kirchhoff rod equations with drag, without inertia, but incorporating a new viscoelastic constitutive relation. This future work may help us better understand slender living bodies in viscous fluids.

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