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Tropical algebra, graph theory, & foreign exchange arbitrage

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TROPICAL ALGEBRA, GRAPH THEORY, & FOREIGN EXCHANGE ARBITRAGE

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Contents

List of Figures

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Abstract

We answer the question, given n currencies and k trades, how can a maximal arbitrage opportunity be found and what is its value? To answer this question, we use techniques from graph theory and employ a max-plus algebra (commonly known as tropical algebra). Further, we show how the tropical eigenvalue of a foreign exchange rate matrix relates to arbitrage among the currencies and can be found algorithmically. We finish by employing time series techniques to study the stability of maximal, high-currency arbitrage opportunities.

Introduction

In this paper, we show how techniques from graph theory and tropical algebra can be used to find maximal arbitrage opportunities in the context of foreign exchange markets. Deeper connections to tropical mathematics, in particular the tropical eigenvalue problem, are also discussed. Further, we employ time series techniques to simulate 40,000 possible ways in which a set of foreign exchange matrices could change. We then analyze the results and isolate some of the more interesting examples.

We do so because riskless profit is a key concept in finance, not only for academics, but also for practitioners who seek out these exploitable opportunities. Financial institutions such as hedge funds and banks design entire strategies around finding riskless profit. Furthermore, trades are made with such high frequency that it is important to have the most efficient technique possible for finding these opportunities.

Ultimately, this thesis works to demonstrate how tools and concepts can be taken from a variety of robust and independent fields to gain insight into another seemingly unrelated field.

Introduction to the Foreign Exchange Market

Suppose you have a trip to Europe scheduled; when in Europe, you will need to use the euro currency to take taxis, buy local goods, etc. To obtain euros, you'll find a currency exchange stand in the airport and *exchange* your US dollars for euros. There is a global financial market where financial institutions and investors do this same thing, just on a larger scale. This market is called the *foreign exchange market*, or forex market. The foreign exchange market is the global financial market where currencies are traded (exchanged) for one another. The forex market is the largest financial market in the world with a daily average trading volume of roughly $3.2 trillion [3]. This large size is due to the fact that the forex market is a global market and is open for trading 24/7. Going back to our example, when exchanging your US dollars for Euros, 1 US dollar would not get you 1 euro - perhaps 1 US dollar could buy 0.9 euros. The value used to determine the conversion is known as the *exchange rate*. The exchange rate in the example would be denoted USD/EUR and would have a value equal to 0.9 (the number of euros that can be bought with one dollar). Therefore, you could trade/exchange 200 US dollars for 180 euros. The exchange rate can be thought of as the price of the foreign currency in terms of the domestic currency. Suppose on your way back into the States you have 30 euros leftover and the exchange rate, EUR/USD, is 1.11. We can therefore buy 33.33 US dollars. Notice that EUR/USD is the reciprocal of USD/EUR.

Suppose instead that EUR/USD = 1.5 and USD/EUR still equals 0.9. Starting with 100 US dollars we could exchange this for 90 euros; we can then exchange this for 135 US dollars. Notice that we end up with 35 more dollars than we started with and took on 0 risk because the exchange rates were not subject to change after we made our first trade. This riskless profit is known as an *arbitrage*. Notice too that there was no fee for making the trades - in finance we say there is an absence of *transaction costs*. One of the most common types of transaction costs when trading is known as the *bid-ask spread*. The bid-ask spread is the difference in the price at which a bank or broker is willing to sell to you, the ask, and the price at which you can sell to them, the bid. For our purposes we will assume there is an absence of all transaction costs, including bid-ask spread.

While this last arbitrage opportunity is obvious, in the real world these opportunities are fleeting, small, and can include any number of currencies. Many financial institutions such as hedge funds dedicate money, time, and human capital towards seeking out these arbitrage opportunities. The buying and selling of those currencies involved in the arbitrage by those institutions will exert either a downward or upward force on the different exchange rates and in turn eliminate the arbitrage. For instance, if exchange rate XY is priced too high (i.e. X buys a disproportionately large amount of Y), the number of people buying currency Y with currency X will increase; these actions will in turn drive down the exchange rate. The buying of Y causes the currency to appreciate relative to X and thus more of X will be required to buy Y. Notice however that we always assume that we can buy from someone in the market and sell to someone in the market at the current prices - why is that?

The answer has to do with the willingness of large banks and brokers to act as *market makers* by buying and selling. The willingness of large banks to buy and sell is motivated by making a profit on the spread and leads the forex market to be quite *liquid*. Roughly speaking, liquidity describes the ease with which something can be bought or sold in the market at a given price. Low bid-ask spreads also contribute to liquidity - a currency pair that is widely traded, i.e. the euro and the US dollar, will have a low bid-ask spread because it is easy for a bank to turn around and then sell or buy the currency you just bought from or sold to them. To illustrate this point, consider that the euro to dollar rate has over 100 different movements, or *ticks*, in one minute during periods of the day with high liquidity [3]. Lesser traded currency pairs will have a higher bid-ask spread for the same reason a car dealer will buy your car for much less than they will sell it for: buyers and sellers are limited and thus at a given price it may be hard to sell. Dealers need that extra wiggle room in price to not only turn a profit but also to be able to lower price if it turns out the market price was lower than initially thought. In liquid markets, there is much less uncertainty as to the true price of something because many people are willing to buy and sell at that price; thus the market agrees on what the price of something should be. The uncertainty in price due to new information that the markets have yet to price in is what leads to arbitrage opportunities. If new information comes in that lowers the yen to euro exchange rate, different people in the market may disagree as to what the new rate should be and what its affects on other rates should be. This uncertainty and increased fluctuation, or *volatility*, is what leads to the presence of arbitrage opportunities.

Volatility is a cornerstone of higher level finance. Volatility is most commonly taken to be the standard deviation of returns for the thing whose volatility we are trying to measure. Volatility can be defined for a stock, a commodity, or currency pair among many other securities. The reason people care so much about volatility is because knowing how prone a security is to large swings can be used to determine risk as well as model the price going forward. Further, *financial derivatives*, securities whose value is *derived* from some underlying security, are a function of, among other things, the underlying security's volatility.

Any market is liable to have arbitrage opportunities because uncertainty and fluctuation will exist regardless. The foreign exchange market is a good candidate for arbitrage opportunities. The frequency, duration, and magnitude of these opportunities on two, three currency trios, (USD, EUR, JPY and USD, EUR, and CHF), was studied by Fenn and Howison where they found approximately 21,000 arbitrage opportunities over a 25 day stretch using tick by tick data. Over 94% of arbitrage opportunities would result in a profit of less than \$100 on a \$1,000,000 trade. Although some opportunities persisted for over 1.5 minutes, 95% of the opportunities were wiped out in under five seconds [3]. Checking for arbitrage on three currencies, *triangular arbitrage*, is easy to check, but arbitrage is not limited to just three currencies; arbitrage can exist on any number of currencies. Further, it is obvious that we should care only about the maximal opportunity. Therefore the question is, if one is given n currencies, and therefore $n^2 - n$ different exchange rates, and allowed to make k trades, what is the largest possible arbitrage opportunity? What are the currencies involved in this arbitrage? Before answering these questions, let us first develop notation for writing down multiple exchange rates. We will do so using a *cross currency matrix* where the $(i, j)^{th}$ entry of the matrix is equal to the exchange rate I/J.

Example 2.1: Consider the following cross currency matrix:

$$\begin{array}{cccc} & USD & EUR & JPY \\ USD & 1 & 0.5 & 100 \\ EUR & 2 & 1 & 133\frac{1}{3} \\ JPY & 0.01 & 0.0075 & 1 \end{array}.$$

Figure 1: Simple Cross Currency Matrix

Notice that the value on the diagonal is one because one unit of a currency is worth exactly one unit of itself. Convince yourself that there does not exist a two way arbitrage. However, notice that it is possible to make an arbitrage on three currencies. Suppose you start by exchanging 1 USD for 100 JPY. We then exchange the 100 JPY for 0.75 EUR. Finally, exchange the 0.75 EUR for 1.5 USD for an arbitrage profit of 0.5 USD. We are effectively finding a series of trades such that when we multiply the exchange rates, we end up with a value greater than one. Convince yourself that the arbitrage value is the same regardless of where you start in the series of trades because multiplication is commutative. We can represent this visually as:

$$
\begin{aligned}
&USD \rightarrow JPY \rightarrow EUR \rightarrow USD \\
&= JPY \rightarrow EUR \rightarrow USD \rightarrow JPY \\
&= EUR \rightarrow USD \rightarrow JPY \rightarrow EUR.
\end{aligned}
$$

The value of 1.5 can be best thought of as a multiplier, meaning that starting with X units of the original currency will result in 1.5X units. The profit is therefore given by 1.5X-X = 0.5X.

Now let us consider this actual cross currency matrix composed of the ask prices for 5 different currencies pulled on October 25, 2016. The upper half of the matrix was taken to be the negative of the lower half to eliminate any two way arbitrages. Notice that these are the *ask* prices so we are assuming no bid-ask spread. These exchange rates were pulled at two minute intervals, and we are assuming the listed price is the price at which we could execute the trade, which is not always true.

Example 2.2: Consider the following cross currency matrix:

	USD	EUR	GBP	JPY	CAD
USD	0	-0.085627	-0.198195	4.64651	0.28835
EUR	0.085627	0	-0.112614	4.73197	0.37396
GBP	0.198195	0.112614	0	4.84457	0.48662
JPY	-4.64651	-4.73197	-4.84457	0	-4.35815
CAD	-0.28835	-0.37396	-0.48662	4.35815	0

Figure 2: Cross Currency Matrix

This matrix is not built from the actual exchange rates themselves but rather the natural log of the exchange rates, and therefore are slightly rounded in the matrix. We stated that our condition for the existence of arbitrage was that there existed a series of trades such that when the exchange rates are multiplied together we get a value greater than one and end on the same currency we started with. If we take the natural log of the exchange rates we can rephrase this as, "there exists an arbitrage if there exists a series of trades such that the exchange rates, when *added*, sum to greater than zero and we end on the same currency we started with." The resulting arbitrage multiplier will be the natural log of the true arbitrage multiplier. However, because arbitrage opportunities in the real world are on the order of 10^{-4}, we get $1 + ln(arb.mult) \approx arb.mult$ because $e^x \approx 1 + x$ for small x.

Although this reformulation of the problem may seem trite, it will become useful when we look to apply techniques from tropical algebra so solve the maximal-arbitrage problem.

Notice with five currencies, US dollar, euro, British pound, Japanese yen, and Canadian dollar, we can look for arbitrage on 3, 4 and 5 currencies. Not only do we look for arbitrage, but again, we are concerned with the maximal arbitrage. The largest triangular arbitrage is given by the sequence

$$GBP \rightarrow CAD \rightarrow JPY \rightarrow GBP.$$

Following this sequence will result in an arbitrage profit of 1.0002 times the amount of pounds that you started with. Again, starting anywhere in this sequence will result in the same multiplier. Notice that we assumed that we held a basket of all currencies and could thus start wherever in the matrix we pleased. Our assumptions of being able to trade instantaneously at the listed price with no transaction costs and holding all currencies is closest to the position of financial institutions.

Returning to Example 2.2, the largest 4 way arbitrage is given by the sequence

$$GBP \to CAD \to USD \to JPY \to GBP.$$

Following this sequence leads to an arbitrage profit of 1.000215 times the amount of currency that we started with. Notice that going backwards through this sequence will result in the same arbitrage. This fact is because XY = -YX. However, swapping the positions of two currencies in the cycle may or may not result in an arbitrage. Now let us consider if an arbitrage on five currencies would be maximal if we are allowed to make five trades (it could be the case that the four way arbitrage is the best we could do and our optimal 5^{th} trade would be no trade). Fortunately for us we do have that, given five trades, the optimal arbitrage involves five currencies. The maximal arbitrage is given by the sequence

$$USD \to GBP \to CAD \to JPY \to EUR \to USD.$$

With this sequence we have an arbitrage multiplier of 1.00024. Notice that the magnitude of these opportunities is quite small even without bid-ask spread considerations and thus a large amount of capital would be required to profit. In this case, $1,000,000 USD would be needed to make a profit of $240 USD.

If asked to find these opportunities, one might try testing every possible sequence of length three, four, and five. This brute force method, apart from being inelegant, is inefficient. The solution to this problem is the main focus of the paper.

Introduction to Graph Theory

When mathematicians wish to visually represent connections between objects, often times a *graph* is used. Graph theory is a widely applicable topic studied both for its own sake and its applicability to networks, logistics, and any subject where the system can be represented as a collection of dots connected by lines.

[6] **Definition 3.1 :** A *graph*, G, is given by a vertex set $V(G) = \{v_1, \cdots, v_n\}$ and an edge set $E(G) = \{e_{i,j}, \cdots, e_{s,t}\}$ where $e_{m,l} \in E(G)$ if and only if v_m and v_l are connected with an edge.

Suppose we wish to represent four airports and the possible flights connecting them as a graph. The vertices are our airports and an edge will connect two vertices if there is a flight connecting them. Note that edges can be traversed in either direction, so if Dulles and JFK are connected with an edge we can take a flight in either direction. Suppose the graph, G, is given by $V(G) = \{v_1, v_2, v_3, v_4\}$ and $E(G) = \{e_{1,2}, e_{1,4}, e_{1,3}, e_{2,3}\}$. We therefore visually represent G by the following graph:

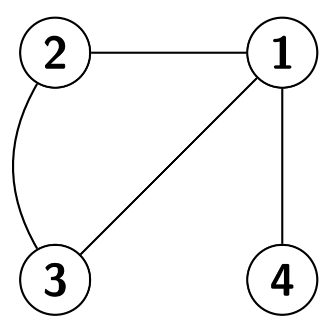


Figure 3: Simple Graph

Suppose we are concerned about the number of possible flights which connect any two airports. If we allow multiple edges to connect two vertices we get a *multigraph*. The number of edges connected to vertex v is known as the *degree* of v, written $deg(v)$. For example, in our example below, $deg(v_1) = 5$.

Example 3.2:

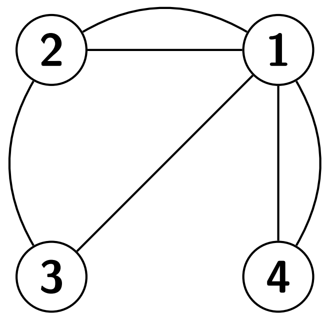


Figure 4: Multigraph

By traveling from one airport to another, we visit a series of vertices $v_1, \cdots, v_n$. This series of vertices is known as a *walk.* If we never visit a vertex (airport) more than once, the walk is called a *path.* If we never traverse the same edge twice in our walk, (i.e. never take the same flight twice), then our walk is called a *trail.* Suppose we have a path $v_1, \cdots, v_n$ and $v_1 = v_n$, we call this a *cycle*; likewise, if we have a trail $v_1, \cdots, v_n$ and $v_1 = v_n$, we call this a *circuit.* A graph with no cycles is called a *tree.* A cycle can be thought of as a round trip where no airport along the way is visited more than once. Similarly, a circuit is a round trip where we never take the same flight. Let's suppose someone wishes to take *every* flight once and only once (i.e. traverse every edge), and wishes to end up where they started, in graph theory this question can be phrased "does there exist an *Eulerian Circuit*?" If there exists an Eulerian Circuit then we simply refer to the graph as *Eulerian.* If there exists a cycle where every vertex is visited, we call this an *Eulerian Cycle.* The question remains, is Example 3.2 Eulerian? Take some time for yourself to try and find an Eulerian Circuit. You'll notice that the graph is not Eulerian, and this fact is due to vertices 1 and 2. Notice that these vertices are of odd degree, this leads us to our first major theorem in graph theory which resulted from solving the Bridges of Konigsberg problem. A graph, G, is Eulerian if and only if every vertex is of even degree [6].

To see why this must be true, consider traversing an Eulerian Circuit. Apart from the starting vertex, every time a vertex is entered through an edge, there must exist a corresponding edge to exit on, therefore every vertex is of even degree. When we consider the starting vertex, we first exit, and every other time we exit, we must have an edge to traverse to get back to the vertex, therefore the starting vertex is also of even degree.

Now consider that when we fly, our flight has a travel time in hours. Suppose we wish to represent this in our graph; we would do so by assigning *weights* to the edges. A graph with weighted edges is, unsurprisingly, known as a *weighted graph.* Suppose the edges are assigned weights and the resulting graph is shown below.

Example 3.3:

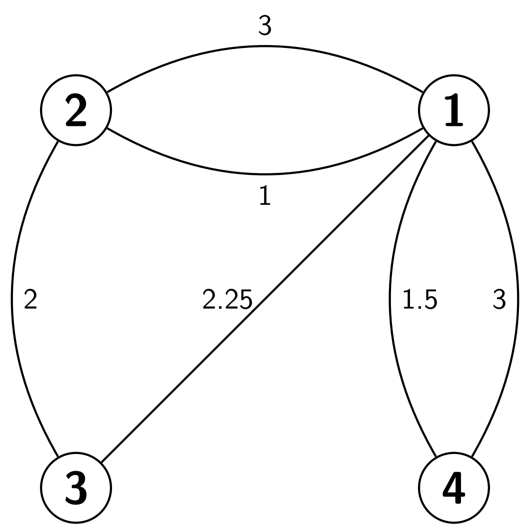


Figure 5: Weighted Graph

We might ask, what is the set of flights such that every airport is visited by at least one flight? In graph theory, we ask about an *edge-cover.* From there we could ask things about the minimal edge-cover. Notice in this case our minimal edge-cover corresponds to the edges with weights 2 and 1.5. Similar to an edge-cover is the notion of a *spanning tree.* In constructing a spanning tree, we look for a collection of edges such that the collection of edges forms a tree and every vertex is included in the tree. Notice the collection of edges with weights 1, 2.25 and 1.5 forms a spanning tree. In a tree, there must exist a path from any vertex on the tree to any other vertex on the tree (this is not the case for an edge-cover). One might also be concerned with the *minimal spanning tree.* In our example, the minimal spanning tree would be given by the edges with weights 2, 1, and 1.5. Obviously most examples are not this simple, and when this is the case we implement either **Kruskal's algorithm** or **Prim's algorithm** to find the minimal spanning tree [2].

Suppose we want to assign every airport a call-sign and for whatever reason we want to use as few call-signs as possible. Our only rule is that no two airports with a flight between them are allowed to have the same call-sign. In graph theoretical terms, "call sign" is called a *color* and we ask what is the minimal number of colors needed to color the graph. We call this number the *chromatic number* of G. In our example, the chromatic number is three (assign 1, 2, and 3 different colors and then assign 4 the color of either 1 or 2). In fact, for any graph where we can write it down without edges crossing (i.e. the graph is *planar*), the chromatic number is at most 4 [6]. This fact is known as the *Four Color Theorem*, the proof of which took almost 200 years and remains a slightly polarizing topic in the mathematical community. This theorem means that any map is 4-colorable and so one only needs four different colors to assure that no two states or countries with the same color share a border. Try and see how a map could be represented as a graph.

Now suppose each flight can only be taken in one direction. If the edges in a graph are assigned a direction, we call the graph a *digraph.* Edges in these graphs can only be traversed in their corresponding direction. If there exists a path from any vertex to any other vertex, then we call the digraph *strongly connected*.

Example 3.4:

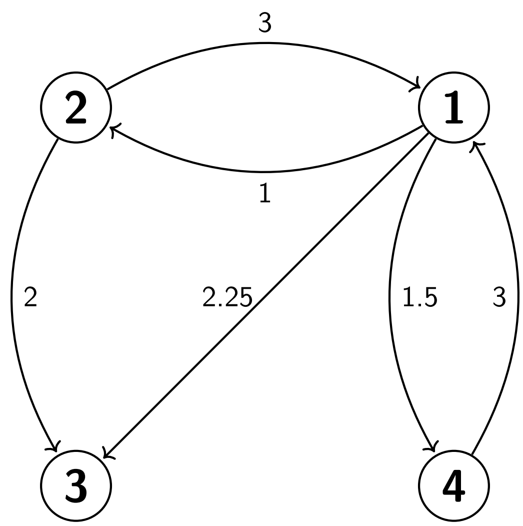
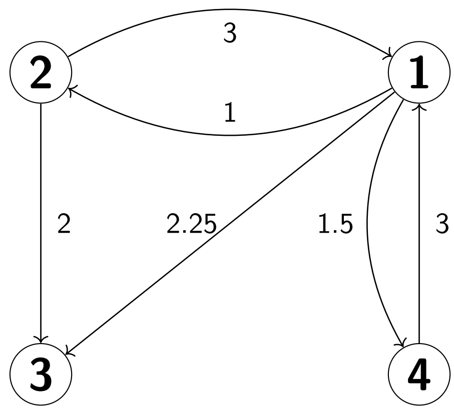


Figure 6: Weighted Digraph

We can continue to ask questions about the properties of digraphs similar to the ones we have been asking. The topics presented were a small sampling of topics in graph theory and the field remains an active area of research for both applied and pure mathematicians.

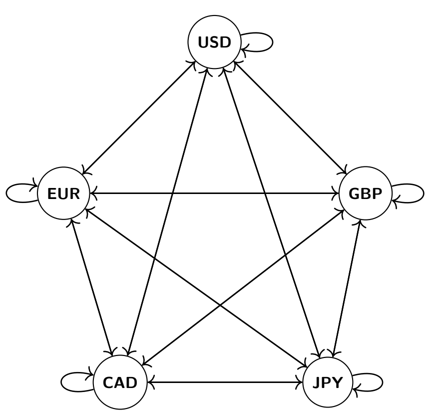
Graphs are not limited to their pictorial representation; they can also be represented by a matrix. For a graph on n vertices, we take an $n \times n$ matrix and we take entry (i, j) to be equal to the weight of the edge originating at vertex i and connecting to vertex j. If there is not an edge going from vertex i to vertex j, we take the edge weight to be zero. We do not allow for multiple edges going from one vertex to another in the same direction. The matrix representation of Example 3.3 would be given by



$$\begin{matrix} 0 & 1 & 2.25 & 1.5 \\ 3 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{matrix}.$$

Figure 7: Ex 3.3 Graph and Corresponding Matrix

Because a graph can be written as a matrix, it stands to reason that a matrix can be viewed as a graph. In particular, we can view the cross currency matrix we developed in the previous chapter as a graph. The graphical representation of Example 2.2 along with its corresponding matrix is shown below. For legibility purposes, the weights are not labeled on the graph. Further, instead of having one edge, for example, going from USD to EUR and another going from EUR to USD, it is represented by a bidirectional arrow. This choice is simply notational.



	USD	EUR	GBP	JPY	CAD
USD	0	-0.085627	-0.198195	4.64651	0.28835
EUR	0.085627	0	-0.112614	4.73197	0.37396
GBP	0.198195	0.112614	0	4.84457	0.48662
JPY	-4.64651	-4.73197	-4.84457	0	-4.35815
CAD	-0.28835	-0.37396	-0.48662	4.35815	0

Figure 8: Foreign Exchange Graphical and Matrix Representation

Our arbitrage sequences of

$$GBP \to CAD \to JPY \to GBP,$$

$$GBP \to CAD \to USD \to JPY \to GBP,$$

and

$$USD \to GBP \to CAD \to JPY \to EUR \to USD$$

are plainly viewed as cycles in a graphical context where we are simply traversing along these weighted and directed edges. Notice that the edge weight going from one currency to another is equal to the natural log of the exchange rate. Further, every currency is connected to itself with edge weight zero.

We can now phrase our question about maximal arbitrage in terms of graph theory. The question now becomes, given that you are allowed to traverse k edges, is there a cycle where the sum of the edge weights is greater than zero? The answer to this question will tell you if there is an arbitrage. More importantly, we want to know *what is the maximal cycle containing k edges*? The sum of these edge weights is the arbitrage multiplier and the vertices traversed in the cycle are the currencies involved in the arbitrage. As a graph theoretical side note, the graph of foreign exchange rates is a *completely connected* graph because every vertex is connected to every other vertex and it is also a *pseudo-graph* because vertices are connected to themselves. Further, even with the edge notation simplified, the graph is not planar, meaning that it is impossible to write the graph down such that no edges cross.

Introduction to Tropical Algebra

In everyday mathematics, the '+' symbol means to add two number together and the '×' symbol means to multiply them. The everyday math studied by grade school students is a special instance of what is known as a *ring*. They study the ring given by $(\mathbb{R}, +, \times)$. We call this structure a ring because addition is commutative and associative, there is an additive identity and an additive inverse, and multiplication is associative and distributive.

However, we need not pigeonhole ourselves to the traditional definitions of addition and multiplication to get many of these same properties. Only in the past couple decades have mathematicians and other scientists employed what is now called the *tropical* approach to mathematics. In tropical algebra, the plus symbol is taken to mean the maximum/minimum of two numbers and the multiplication symbol is taken to mean standard addition. Typically the symbols $\oplus$ and $\otimes$ are used to denote tropical plus and tropical times. That is, we define $a \oplus b = \max\{a, b\}$ and $a \otimes b = a + b$ [4].

If we were to use the definitions presented, we would call this a *max-plus* algebra because of our choice that $a \oplus b = \max\{a, b\}$; defining the result as $a \oplus b = \min\{a, b\}$ would be referred to as a *min-plus* algebra. Let us now examine the algebra to see if it can indeed be called a ring.

Let us first mention the fact that in the max-plus world, we equip ourselves with an additive identity, $-\infty$, where $a + -\infty = a$ for all $a \in \mathbb{R}$. The additive identity further has the property that $-\infty \otimes a = -\infty$. In the min-plus universe, our additive identity would be ∞. These operators share many key properties with their standard cousins. Notice that tropical addition is commutative over $\mathbb{R}$ because for every a, b in $\mathbb{R}$ we have that $max\{a, b\} = max\{b, a\}$. It is also associative because $max\{max\{a, b\}, c\} = max\{max\{a, c\}, b\}$. Tropical multiplication is commutative and associative over $\mathbb{R}$ because tropical multiplication is just standard addition in disguise. These last properties were fairly obvious, but is it the case that tropical multiplication distributes over tropical addition? Keep in mind that order of operations still applies. Let us consider $(a \oplus b) \otimes c$. Without loss of generality, let us suppose that $a < b$. Then we have

$$a < b \ \Rightarrow a + c < b + c \ \Rightarrow max\{a, b\} + c = max\{a + c, b + c\}.$$

We therefore get that tropical multiplication is distributive (notice that the same properties would hold in a min-plus algebra).

While so far tropical algebra has preserved several necessary ring properties, let us now consider if there exists a additive inverse; that is, is it the case that for every element, there is a corresponding element such that when added together we end up with the additive identity? The answer is no; notice that minus infinity is by definition smaller than every element and therefore can never be returned by the *max* function (except in the case where all arguments are minus infinity). A similar argument holds in the min-plus case. The lack of an additive inverse for every element precludes the algebra from being a ring and as a consolation prize of sorts, mathematicians call such structures semi-rings. Therefore, tropical algebra is a semi-ring given by $(\mathbb{R}^{\max}, \oplus, \otimes)$ where $\mathbb{R}^{\max} = \mathbb{R} \cup \{-\infty\}$ [4].

Interesting to note is that in a max-plus algebra, subtraction does not exist and instead we speak of multiplying by a negative number to accomplish the same result. Further, polynomials become piece-wise defined. But why though do mathematicians go through the trouble of redefining convention, and is tropical mathematics only studied and employed in a pure math setting? As it turns out, tropical mathematics has become both a useful and indeed crucial tool for areas including economics, computer science, and physics. In fact, when the Bank of England approached renowned economist Paul Klemperer to help develop a technique for efficiently and most beneficially auctioning off capital, the solution he came to is entirely tropical [7][8]. Part the tropical framework's allure is that it can allow for a reduction in a problem's computational complexity.

Just as in the standard algebra, we can use tropical operations in the context of matrices. Consider the following example of matrix addition in the tropical context.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \oplus \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} max\{a,e\} & max\{b,f\} \\ max\{c,g\} & max\{d,h\} \end{bmatrix}.$$

Using real numbers for example, we would have

$$\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 5 & -\infty \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 6 & 0 \end{bmatrix}.$$

In the tropical algebra, addition of two $n \times n$ matrices is defined in the same fashion as usual where $c_{i,j} = a_{i,j} \oplus b_{i,j}$. Now let's do an example of matrix multiplication under a tropical algebra.

$$\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 5 & -\infty \\ 6 & -3 \end{bmatrix} = \begin{bmatrix} (2 \otimes 5) \oplus (4 \otimes 6) & (2 \otimes -\infty) \oplus (4 \otimes -3) \\ (1 \otimes 5) \oplus (0 \otimes 6) & (1 \otimes -\infty) \oplus (0 \otimes -3) \end{bmatrix}$$

$$= \begin{bmatrix} 7 \oplus 10 & -\infty \oplus 1 \\ 6 \oplus 6 & -\infty \oplus 0 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ 6 & 0 \end{bmatrix}.$$

When we wish to write a matrix to a given power, n, we write

$$\begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix}^{\otimes n}.$$

Let us compute tropical powers of this matrix.

$$\begin{bmatrix}2 & 4\\1 & 0\end{bmatrix}^{\otimes 2} = \begin{bmatrix}2 & 4\\1 & 0\end{bmatrix} \otimes \begin{bmatrix}2 & 4\\1 & 0\end{bmatrix} = \begin{bmatrix}\max\{2+2,\ 4+1\} & \max\{2+4,\ 4+0\}\\ \max\{1+2, 0+1\} & \max\{1+4,\ 0+0\}\end{bmatrix} = \begin{bmatrix}5 & 6\\3 & 5\end{bmatrix}.$$

We leave it to the reader to check that the following is correct.

$$\begin{bmatrix}2 & 4\\1 & 0\end{bmatrix}^{\otimes 3} = \begin{bmatrix}7 & 9\\6 & 7\end{bmatrix}.$$

Tropical multiplication of a matrix by a scalar works exactly as one would think it should. For a given real number r, we get

$$r \otimes \begin{bmatrix}a & b\\c & d\end{bmatrix} = \begin{bmatrix} r\otimes a & r\otimes b\\ r\otimes c & r\otimes d\end{bmatrix}.$$

In a standard algebra over $\mathbb{R}^{n\times n}$ we are often concerned with finding the *eigenvalues* of a matrix along with the corresponding *eigenvectors.*

Definition 4.1: An eigenvalue-eigenvector pair λ and x for a given matrix A satisfy the equation $Ax = \lambda x$ where $\lambda \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, and $x \in \mathbb{R}^n$.

It is a reasonable question to ask if we have tropical eigenvalues and eigenvectors, and indeed we do. First, let us think about what it would mean for a matrix to have a tropical eigenvalue and eigenvector.

Consider the following matrix with a tropical eigenvalue and eigenvector as follows:

$$\begin{bmatrix}a & b\\c & d\end{bmatrix} \otimes \mathbf{x} = \lambda \otimes \mathbf{x}$$

where

$$\mathbf{x} = \begin{bmatrix}r\\s\end{bmatrix}.$$

It follows then that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} r \\ s \end{bmatrix} = \lambda \otimes \begin{bmatrix} r \\ s \end{bmatrix}.$$

Therefore, if we were to write this in a standard algebra, we would find

$$\begin{bmatrix} \max\{a+r, b+s\} \\ \max\{c+r, d+s\} \end{bmatrix} = \begin{bmatrix} r+\lambda \\ s+\lambda \end{bmatrix}.$$

It is now clear that r, s and λ must satisfy the conditions that $\max\{a+r, b+s\} = r+\lambda$ and $\max\{c+r, d+s\} = s+\lambda$. Eigenvector-eigenvalue pairs are widely applicable and useful for repeatedly multiplying by a matrix. We now know what it would mean for tropical eigenvalues and eigenvectors to exist, but the question remains, how does one find them?

The problem of trying to find a tropical eigenvalue for some matrix A is equivalent to trying to find the maximal normalized weight across all cycles in the graphical representation ($G(A)$), assuming that the graph is strongly connected. We define the *normalized weight* of a cycle to be the sum of the edge weights involved in a cycle divided by the number of edges.

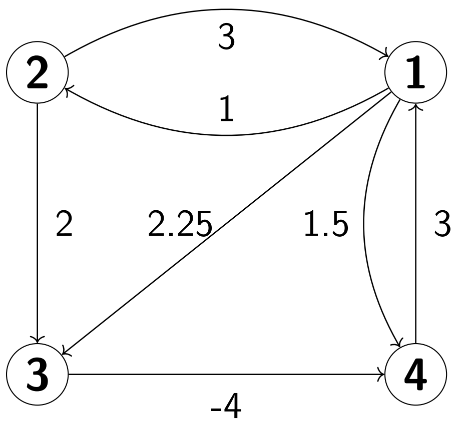
[5] Theorem 4.1: *If $A \in \mathbb{R}^{n\times n}$ with $G(A)$ strongly connected, then A has a unique tropical eigenvalue equal to the maximal normalized weight across all directed cycles in $G(A)$.*

We do not, however, need to check very possible cycle in the graph. In 1978, Richard Karp published an algorithm for finding the maximal normalized weight across all cycles. Before showing Karp's algorithm, we note that the identity matrix in the max-plus tropical algebra, B, where $b_{ij} = 0$ if $i = j$ and $-\infty$ otherwise. Now we are ready for Karp's algorithm [1][9]:

- For $A \in \mathbb{R}^{n\times n}$ arbitrarily choose some column, b_j, of the $n \times n$ identity matrix; set $b_j = v(0)$.

- Compute $v(k) = A \otimes v(k-1)$ for $k = 1, 2, \cdots, n$
- Compute $\lambda = \max_{i=1,\cdots,n}\left[\min_{k=0,\cdots,n-1}\left[\frac{v_i(n)-v_i(k)}{n-k}\right]\right]$

Before progressing through our foreign exchange example, let us revisit our favorite network of airports and see how Karp's algorithm would be implemented. Because we require our graph be strongly connected, we have to add a route going from 3. Let's choose it to have weight -4 and to travel to airport 4.



$$\begin{bmatrix} -\infty & 1 & 2.25 & 1.5 \\ 3 & -\infty & 2 & -\infty \\ -\infty & -\infty & -\infty & -4 \\ 3 & -\infty & -\infty & -\infty \end{bmatrix}$$

Figure 9: Weighted Digraph and Tropical Matrix Representation

Take note that in a tropical algebra, instead of taking an edge weight to be zero between to vertices if there does not exist an edge, we take the edge weight to be $-\infty$, the additive identity under a tropical algebra. Let's implement Karp's algorithm and let's arbitrarily take our $v(0)$ to be the 3^{rd} column in the identity matrix. Our first step is therefore:

$$v(1) = A \otimes v(0) = \begin{bmatrix} -\infty & 1 & 2.25 & 1.5 \\ 3 & -\infty & 2 & -\infty \\ -\infty & -\infty & -\infty & -4 \\ 3 & -\infty & -\infty & -\infty \end{bmatrix} \otimes \begin{bmatrix} -\infty \\ -\infty \\ 0 \\ -\infty \end{bmatrix} =$$

$$\begin{bmatrix} (-\infty \otimes -\infty) \oplus (1 \otimes -\infty) \oplus (2.25 \otimes 0) \oplus (1.5 \otimes -\infty) \\ (3 \otimes -\infty) \oplus (-\infty \otimes -\infty) \oplus (2 \otimes 0) \oplus (-\infty \otimes -\infty) \\ (-\infty \otimes -\infty) \oplus (-\infty \otimes -\infty) \oplus (-\infty \otimes 0) \oplus (-4 \otimes -\infty) \\ (3 \otimes -\infty) \oplus (-\infty \otimes -\infty) \oplus (-\infty \otimes 0) \oplus (-\infty \otimes -\infty) \end{bmatrix} = \begin{bmatrix} 2.25 \\ 2 \\ -\infty \\ -\infty \end{bmatrix}$$

It should not be surprising that by multiplying by the 3^{rd} column of the identity matrix, we find ourselves left with the 3^{rd} column of A. Notice, this column of A represents all possible paths of length 1 that lead to vertex 3.

$$v(2) = A \otimes v(1) = \begin{bmatrix} -\infty & 1 & 2.25 & 1.5 \\ 3 & -\infty & 2 & -\infty \\ -\infty & -\infty & -\infty & -4 \\ 3 & -\infty & -\infty & -\infty \end{bmatrix} \otimes \begin{bmatrix} 2.25 \\ 2 \\ -\infty \\ -\infty \end{bmatrix} =$$

$$\begin{bmatrix} (-\infty \otimes 2.25) \oplus (1 \otimes 2) \oplus (2.25 \otimes -\infty) \oplus (1.5 \otimes -\infty) \\ (3 \otimes 2.25) \oplus (-\infty \otimes 2) \oplus (2 \otimes -\infty) \oplus (-\infty \otimes -\infty) \\ (-\infty \otimes 2.25) \oplus (-\infty \otimes 2) \oplus (-\infty \otimes -\infty) \oplus (-4 \otimes -\infty) \\ (3 \otimes 2.25) \oplus (-\infty \otimes 2) \oplus (-\infty \otimes -\infty) \oplus (-\infty \otimes -\infty) \end{bmatrix} = \begin{bmatrix} 3 \\ 5.25 \\ -\infty \\ 5.25 \end{bmatrix}$$

Notice the n^{th} row of this vector is the maximal weight path of length two starting at vertex n and finishing at vertex 3. For the penultimate iteration, we find

$$v(3) = A \otimes v(2) = \begin{bmatrix} -\infty & 1 & 2.25 & 1.5 \\ 3 & -\infty & 2 & -\infty \\ -\infty & -\infty & -\infty & -4 \\ 3 & -\infty & -\infty & -\infty \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 5.25 \\ -\infty \\ 5.25 \end{bmatrix} =$$

$$\begin{bmatrix} (-\infty \otimes 3) \oplus (1 \otimes 5.25) \oplus (2.25 \otimes -\infty) \oplus (1.5 \otimes 5.25) \\ (3 \otimes 3) \oplus (-\infty \otimes 5.25) \oplus (2 \otimes -\infty) \oplus (-\infty \otimes 5.25) \\ (-\infty \otimes 3) \oplus (-\infty \otimes 5.25) \oplus (-\infty \otimes -\infty) \oplus (-4 \otimes 5.25) \\ (3 \otimes 3) \oplus (-\infty \otimes 5.25) \oplus (-\infty \otimes -\infty) \oplus (-\infty \otimes 5.25) \end{bmatrix} = \begin{bmatrix} \max\{6.25, 6.75\} \\ 6 \\ 1.25 \\ 6 \end{bmatrix}.$$

At this stage, let's more closely examine what exactly the algorithm is doing. Notice that we are choosing the maximum between 6.25 and 6.75. These weights correspond to the options of traveling $1 \to 2 \to 1 \to 3$ or going $1 \to 4 \to 1 \to 3$. Our second entry has gone from 5.25

to 6 because now instead of going $4 \to 1 \to 3$ we can travel $4 \to 1 \to 2 \to 3$. Because we now have paths of length three, we are able to start and end at 3 by going $3 \to 4 \to 1 \to 3$. For the last iteration, we find

$$v(4) = A \otimes v(3) = \begin{bmatrix} -\infty & 1 & 2.25 & 1.5 \\ 3 & -\infty & 2 & -\infty \\ -\infty & -\infty & -\infty & -4 \\ 3 & -\infty & -\infty & -\infty \end{bmatrix} \otimes \begin{bmatrix} 6.75 \\ 6 \\ 1.25 \\ 6 \end{bmatrix} =$$

$$\begin{bmatrix} (-\infty \otimes 6.75) \oplus (1 \otimes 6) \oplus (2.25 \otimes 1.25) \oplus (1.5 \otimes 6) \\ (3 \otimes 6.75) \oplus (-\infty \otimes 6) \oplus (2 \otimes 1.25) \oplus (-\infty \otimes 6) \\ (-\infty \otimes 6.75) \oplus (-\infty \otimes 6) \oplus (-\infty \otimes 1.25) \oplus (-4 \otimes 6) \\ (3 \otimes 6.75) \oplus (-\infty \otimes 6) \oplus (-\infty \otimes 1.25) \oplus (-\infty \otimes 6) \end{bmatrix} = \begin{bmatrix} 7.5 \\ 9.75 \\ 2 \\ 9.75 \end{bmatrix}.$$

Let us now write down again all of the $v(k)$'s.

$$v(0) = \begin{bmatrix} -\infty \\ -\infty \\ 0 \\ -\infty \end{bmatrix};\ v(1) = \begin{bmatrix} 2.25 \\ 2 \\ -\infty \\ -\infty \end{bmatrix};\ v(2) = \begin{bmatrix} 3 \\ 5.25 \\ -\infty \\ 5.25 \end{bmatrix};\ v(3) = \begin{bmatrix} 6.75 \\ 6 \\ 1.25 \\ 6 \end{bmatrix};\ v(4) = \begin{bmatrix} 7.5 \\ 9.75 \\ 2 \\ 9.75 \end{bmatrix}.$$

The final part of the algorithm is to choose the maximum of the following minimums.

$$\min\left[\frac{7.5+\infty}{4-0}, \frac{7.5-2.25}{4-1}, \frac{7.5-3}{4-2}, \frac{7.5-6.75}{4-3}\right] = .75$$

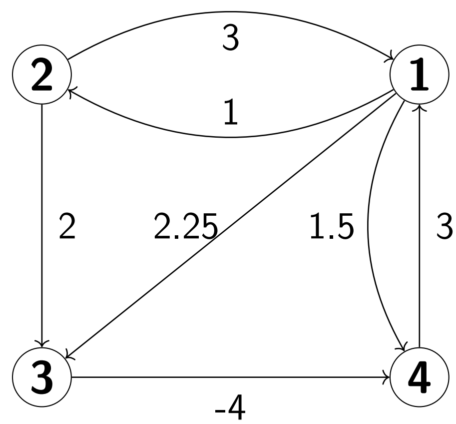
$$\min\left[\frac{9.75+\infty}{4-0}, \frac{9.75-2}{4-1}, \frac{9.75-5.25}{4-2}, \frac{9.75-6}{4-3}\right] = 2.25$$

$$\min\left[\frac{2-0}{4-0}, \frac{2+\infty}{4-1}, \frac{2+\infty}{4-2}, \frac{2-1.25}{4-3}\right] = .5$$

$$\min\left[\frac{9.75+\infty}{4-0}, \frac{9.75+\infty}{4-1}, \frac{9.75-5.25}{4-2}, \frac{9.75-6}{4-3}\right] = 2.25.$$

The maximum of these values is of course 2.25. The claim, then, is that 2.25 is the maximal normalized weight across all cycles in G and is also *the* eigenvalue for its matrix. Looking at

G again, we can see it is obvious that traversing the cycle between vertices 1 and 4 will give us the largest "weight-bang" for our "edge-traversing-buck" and that the normalized weight is 2.25.

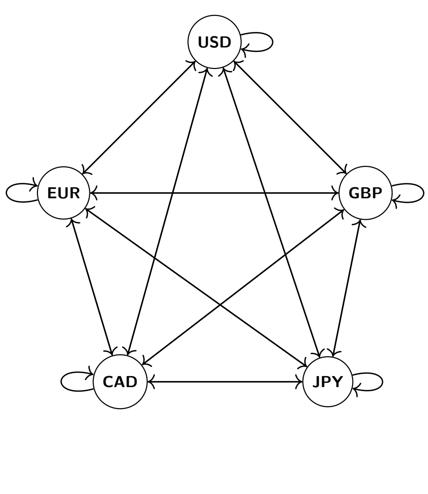


Now consider if we were to implement Karp's algorithm on a matrix of foreign exchange rates. We would be given the maximal *normalized* cycle weight. Meaning, Karp's algorithm can tell us the maximal (average) arbitrage multiplier per trade. However, it does not tell us the vertices involved in this cycle. Further, the number is the maximal *average arbitrage per trade* and thus we are unsure of our maximal arbitrage value; it may be a multiple of the number produced by Karp's algorithm, or it could be an entirely different number. One can easily imagine a scenario where the maximal arbitrage per trade is on a cycle of length three but the maximal arbitrage possible is on a cycle of length four. With which should we concern ourselves? That question aside, it remains to be answered, how do we detect the largest arbitrage possible? Again, we look to tropical algebra.

Theorem 4.2: *Given a matrix, $A \in \mathbb{R}_{\max}^{n\times n}$, the maximal path weight of length k in $G(A)$ going from vertex i to vertex j is given by $a_{i,j}^{\otimes k}$ where $A^{\otimes k}$ is the k^{th} tropical power of A.*

PROOF: We proceed by induction on k. The first power is trivial so let us take $k = 2$ as our base case. By definition, $a_{i,j}^{\otimes 2} = \max_{m=1,\cdots,n}\{a_{i,m}+a_{m,j}\}$. Notice that the set of all possible paths of length two is given by $\{a_i \to a_m \to a_j\}$ for $m = 1, \cdots, n$. The set of all length two path weights would therefore be given by $\{a_{i,m} + a_{m,j}\}$ for $m = 1, \cdots, n$. Therefore $a_{i,j}^{\otimes 2}$ is the maximal length two path weight. Now assume that up to $k = N$, $a_{i,j}^{\otimes k}$ is the maximal path weight of length k going from vertex i to j. By definition, $a_{i,j}^{\otimes N} = \max_{m=1,\cdots,n}\{a_{i,m}^{\otimes N-1} + a_{m,j}\}$. By the induction hypothesis, $a_{i,m}^{\otimes N-1}$ is the maximal path weight of length $N-1$ going from vertex i to vertex m. Let us just consider the $N-1$ steps in the maximal path before we travel to vertex j; we finish this path on some m^* and from there, we travel to j. If the path on which we traveled to m^* was not maximal and was of weight ω, then we could instead travel the path of weight $a_{i,m^*}^{\otimes N-1}$ and our new path would be of greater weight because $a_{i,m^*}^{\otimes N-1} + a_{m^*,j} > \omega + a_{m^*,j}$. Therefore we only need to consider the maximal paths of length $N-1$ leading up to our final edge traversing. The possible path weights are therefore given by the set $\{a_{i,m}^{\otimes N-1} + a_{m,j}\}$ for $m = 1, \cdots, n$. Therefore, by definition, $a_{i,j}^{\otimes N}$ is the maximal path weight of length N going from vertex i to vertex j. ■

Before presenting the corollary to this theorem, which is our main result, let us return to our arbitrage example given by the following graph and corresponding matrix.



	USD	EUR	GBP	JPY	CAD
USD	0	-0.08556	-0.1981	4.6470	0.2886
EUR	0.08563	0	-0.1124	4.7322	0.3741
GBP	0.1982	0.1126	0	4.8450	0.4867
JPY	-4.6465	-4.7320	-4.8446	0	-4.3581
CAD	-0.2884	-0.3740	-0.4866	4.3582	0

.

We are interested in the tropical powers of the matrix given by

$$A^{\otimes 3} = \begin{bmatrix} 0.00017 & -0.085443 & -0.198057 & 4.646583 & 0.288564 \\ 0.085658 & 0.00017 & -0.11243 & 4.732163 & 0.374053 \\ 0.198282 & 0.112807 & \mathbf{0.000207} & 4.844785 & 0.486621 \\ -4.646298 & -4.731916 & -4.844538 & \mathbf{0.000207} & -4.357949 \\ -0.288186 & -0.3738 & -0.486406 & 4.358174 & \mathbf{0.000207} \end{bmatrix};$$

$$A^{\otimes 4} = \begin{bmatrix} \mathbf{0.000215} & -0.085387 & -0.197988 & 4.64672 & 0.288564 \\ 0.085797 & 0.000193 & -0.112407 & 4.73221 & 0.374191 \\ 0.198435 & 0.112821 & \mathbf{0.00022} & 4.844795 & 0.486828 \\ -4.646289 & -4.731763 & -4.844363 & \mathbf{0.00022} & -4.357917 \\ -0.288142 & -0.373759 & -0.486381 & 4.358364 & \mathbf{0.00022} \end{bmatrix};$$

$$A^{\otimes 5} = \begin{bmatrix} \mathbf{0.00024} & -0.08525 & -0.19785 & 4.646728 & 0.288633 \\ 0.085842 & \mathbf{0.00024} & -0.112361 & 4.732348 & 0.374214 \\ 0.198479 & 0.112862 & \mathbf{0.00024} & 4.844985 & 0.486836 \\ -4.646136 & -4.731749 & -4.844355 & \mathbf{0.00024} & -4.357742 \\ -0.288132 & -0.373606 & -0.486206 & 4.358371 & \mathbf{0.00024} \end{bmatrix}.$$

Remember, earlier we found the largest arbitrage multiplier on three currencies was 1.0002, on four currencies was 1.00022, and on five currencies was 1.00024.

Notice that in each of these instances, we get that the largest element on the diagonal describes the arbitrage multiplier, and the currencies involved in producing it (because each column corresponds to a currency). What are the other elements on the diagonal describing then? Well, this value is the largest arbitrage that includes that currency using k trades; this reason is precisely why those 3, 4, and 5 currencies all have the same value. The currencies are all involved in the *maximal* cycle and thus it must be that case that the maximal cycle they are involved in has the same weight for each of them.

How do these values relate the Karp's algorithm? If these are all maximal arbitrages, and Karp's algorithm produces only one number, then how can both be correct? Remember, Karp's algorithm is concerned with the maximal *normalized* cycle weight, not the maximal possible cycle weight. If we implement Karp's algorithm, we find it returns a value of 0.000069. We now note, that

$$\frac{0.000207}{3} = 0.000069,$$

$$\frac{0.000215}{4} = 0.000054,$$

$$\frac{0.00024}{5} = 0.000048.$$

Karp's algorithm has served us well and has indeed found the cycle with the largest normalized weight, across all possible cycle lengths (we exclude cycles of length two because, as mentioned before, there does not exist a two way arbitrage).

We have only shown one example; in the interest of further persuasion and illustration, consider the following cross currency matrix, A, that occurred later that same day. In this example, we did allow for the possibility of two way arbitrage.

	USD	EUR	GBP	JPY	CAD
USD	0	-0.085122	-0.198085	4.64612	0.289006
EUR	0.085259	0	-0.112978	4.731362	0.374242
GBP	0.198276	0.11306	0	4.844407	0.48729
JPY	-4.646096	-4.731198	-4.844316	0	-4.357021
CAD	-0.289016	-0.374111	-0.487109	4.357208	0

Figure 10: Cross currency matrix with two way arbitrage

Let us compute the tropical powers as well as Karp's algorithm.

$$A^{\otimes 3} = \begin{bmatrix} 0.000235 & -0.084875 & -0.197893 & 4.646414 & 0.2893 \\ 0.085451 & 0.000253 & -0.112768 & 4.731582 & 0.374465 \\ 0.198469 & 0.113301 & 0.000276 & 4.844599 & 0.487482 \\ -4.645854 & -4.731011 & -4.844023 & 0.000276 & -4.356835 \\ -0.288729 & -0.373899 & -0.486917 & 4.357395 & 0.000276 \end{bmatrix};$$

$$A^{\otimes 4} = \begin{bmatrix} 0.000383 & -0.084783 & -0.197808 & 4.646514 & 0.289397 \\ 0.085512 & 0.000383 & -0.112633 & 4.731674 & 0.37456 \\ 0.198561 & 0.1134 & 0.000383 & 4.844691 & 0.487577 \\ -4.645746 & -4.730921 & -4.843939 & 0.000383 & -4.356732 \\ -0.28864 & -0.373802 & -0.486814 & 4.35749 & 0.000373 \end{bmatrix};$$

$$A^{\otimes 5} = \begin{bmatrix} 0.000475 & -0.084684 & -0.197701 & 4.646606 & 0.289492 \\ 0.085643 & 0.000475 & -0.112548 & 4.731773 & 0.374657 \\ 0.19866 & 0.113492 & 0.000475 & 4.844791 & 0.487674 \\ -4.645661 & -4.730814 & -4.843831 & 0.000475 & -4.356638 \\ -0.288538 & -0.373708 & -0.486725 & 4.357592 & 0.000475 \end{bmatrix}.$$

From the diagonals of the matrices we are able to again see the maximal opportunities and the currencies involved. Computing Karp's algorithm yields a value of 0.0000959. Checking again as we did in the previous example, we find

$$\frac{0.000277}{3} = 0.0000921,$$

$$\frac{0.000384}{4} = 0.0000959,$$

$$\frac{0.000475}{5} = 0.0000951.$$

Again, Karp's algorithm agrees with our results from taking the matrix to different tropical powers. Now, let us examine the following corollaries to Theorem 4.2.

Corollary 4.3: *Given a cross-currency matrix with n currencies, the maximal arbitrage using k trades and involving currency i is given by $a_{i,i}^{\otimes k}$.*

Corollary 4.4: *Given a cross-currency matrix with n currencies, the maximal arbitrage using k trades is given by $\max_{l=1,\cdots,n}\{a_{l,l}^{\otimes k}\}$. If k distinct currencies $c_1, c_2, \cdots, c_k$ are involved in the maximal cycle then $a_{c_1,c_1}^{\otimes k} = a_{c_2,c_2}^{\otimes k} = \cdots = a_{c_k,c_k}^{\otimes k}$.*

To see why Corollary 4.3 falls out from Theorem 4.2, consider that when we examine $a_{i,i}^{\otimes k}$ we are getting the maximal value of a path starting and ending at currency i; paths that start and end on the same nodes are, by definition, cycles. Further, because the cycle weights are foreign exchange rates, we interpret a positive cycle weight as an arbitrage. Corollary 4.4 is immediate from Corollary 4.3.

In our example with no two way arbitrage, we saw that the maximal normalized cycle weight decreased as the cycles got longer. For our example with two way arbitrage, this fact was not true. This observation begs the question, is it ever the case that we could have tiny arbitrage values for short cycles and considerably larger arbitrage values for longer cycles, all in an environment with no two way arbitrage?

In the next chapter, we show the results of simulating 40,000 possible changes to the currency matrix and produce instances in which there was no two way arbitrage, and the longer cycle arbitrages were considerably larger than the triangular arbitrage.

Vector Autoregressions & Impulse Functions

Foreign exchange rates are a system where different values are correlated and tied to several other values. For instance, if the US dollar appreciates relative to the euro, assuming adjustments do not happen simultaneously, other exchange rates need to adjust in order to eliminate the possibility of three way arbitrage. When those other currencies adjust, this itself causes other foreign exchange rates to change in order to eliminate arbitrage. Further, it is often the case that historical values are used to predict future values, as what just happened/has been happening is our best guess as to what will happen next. When we have systems where several variables are tied together and historic values could be assumed to influence future values, often times a *vector autoregression* is employed to model the system. For 10 foreign exchange rates, $f_1, \cdots, f_{10}$, a first order vector autogression model is of the form [11]

$$\begin{bmatrix} f_{1,t} \\ f_{2,t} \\ \vdots \\ f_{10,t} \end{bmatrix} = \begin{bmatrix} \phi_{0,f_1} \\ \phi_{0,f_2} \\ \vdots \\ \phi_{0,f_{10}} \end{bmatrix} + \begin{bmatrix} \phi_{1,f_1} & \phi_{2,f_1} & \cdots & \phi_{10,f_1} \\ \phi_{1,f_2} & \phi_{2,f_2} & \cdots & \phi_{10,f_2} \\ \vdots & & & \\ \phi_{1,f_{10}} & \phi_{2,f_{10}} & \cdots & \phi_{10,f_{10}} \end{bmatrix} \begin{bmatrix} f_{1,t-1} \\ f_{2,t-1} \\ \vdots \\ f_{10,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{f_1,t} \\ \epsilon_{f_2,t} \\ \vdots \\ \epsilon_{f_{10},t} \end{bmatrix}.$$

The matrix on the left represents the exchange rates at time t. The first matrix on the left is the matrix of constants. The next matrix is the coefficient matrix for the foreign exchange rates at time $t-1$, and it is of course multiplied by the matrix of foreign exchange rates at time $t-1$. The final matrix is the error matrix for the state of the system at time t. We call this vector autoregression (VAR) a *first order* model because we only lag time by one unit. For more on vector autoregressions, see [11] and for applications to foreign exchange rate modeling, see [10].

Once a VAR model is generated, an impulse response function can be used to see how the system is predicted to change through time given a one standard deviation "shock" to the error term of one of the variables [11]. VAR models and impulse functions are powerful tools in time series, however, because we simply need to them to see many different ways in which the system could plausibly change, we will not delve into the more technical aspects.

We estimated a VAR model where we assumed the USDEUR exchange rate was the main driver; a shock to this major currency pair would likely have a strong effect on the others. In our model, we included nine other exchange rates. We only needed nine because if we assume no two way arbitrage, then the change to XY will be exactly -YX (because we have taken the natural log of exchange rates). Therefore, because XY and YX vary linearly with each other, one was excluded due to multicollinearity concerns. From there, a fifth order VAR model was estimated and a shock to the USDEUR exchange rate was simulated by an impulse response function. The impulse response function returned the estimated change in the other foreign exchange rates.

Because a VAR model is linear, the responses to a shock will scale as the shock itself is scaled. After we estimated the impulse function, we scaled the shock (and the responses to it) 40,000 different times by multiplying the shock and responses by the same random variable with mean one and standard deviation four. The scaling factor was randomly multiplied by either one or negative one. Then, the responses in the different currency pairs were allowed to "wiggle." By wiggle, we mean to say that the change in, for instance, JPYCAD, was taken to be the predicted change (including the scaling factor) times z where z was distributed normally with mean 1 and standard deviation 1. Note that a different z value was generated for each simulation and each currency pair was multiplied by its own z.

These changes were then added to our original, no two way arbitrage matrix. The following are interesting examples of ways in which the matrix changed. Note that the way in which we simulated the changes did not allow for us to induce a two way arbitrage. As a reminder, our initial matrix was given by

	USD	EUR	GBP	JPY	CAD
USD	0	-0.085627	-0.198195	4.64651	0.28835
EUR	0.085627	0	-0.112614	4.73197	0.37396
GBP	0.198195	0.112614	0	4.84457	0.48662
JPY	-4.64651	-4.73197	-4.84457	0	-4.35815
CAD	-0.28835	-0.37396	-0.48662	4.35815	0

.

This matrix had a triangular arbitrage of 1.0002, a four way arbitrage of 1.00022, and a five way arbitrage of 1.00024.

Now consider the following matrix and its tropical powers.

Example 5.1:

$$A_1 = \begin{bmatrix} 0 & -0.08538 & -0.197944 & 4.646478 & 0.288486 \\ 0.08538 & 0 & -0.112648 & 4.732008 & 0.374006 \\ 0.197944 & 0.112648 & 0 & 4.844532 & 0.486533 \\ -4.646478 & -4.732008 & -4.844532 & 0 & -4.358143 \\ -0.288486 & -0.374006 & -0.486533 & 4.358143 & 0 \end{bmatrix};$$

$$A_1^{\otimes 3} = \begin{bmatrix} 0.000151 & -0.085296 & -0.197903 & 4.646769 & 0.28871 \\ 0.085671 & 0.00015 & -0.112383 & 4.732149 & 0.374016 \\ 0.198198 & 0.112674 & 0.000144 & 4.844797 & 0.486654 \\ -4.646478 & -4.731774 & -4.844422 & 0.000151 & -4.357852 \\ -0.288335 & -0.373715 & -0.486279 & 4.358143 & 0.000151 \end{bmatrix};$$

$$A_1^{\otimes 4} = \begin{bmatrix} 0.000291 & -0.085229 & -0.197763 & 4.646853 & 0.28871 \\ 0.085671 & 0.000291 & -0.112273 & 4.732159 & 0.374157 \\ 0.198319 & 0.112818 & 0.000265 & 4.844797 & 0.486684 \\ -4.646327 & -4.731774 & -4.844381 & 0.000291 & -4.357768 \\ -0.288335 & -0.373631 & -0.486279 & 4.358294 & 0.000291 \end{bmatrix};$$

$$A_1^{\otimes 5} = \begin{bmatrix} 0.000375 & -0.085089 & -0.197653 & 4.646853 & 0.288777 \\ 0.085681 & 0.000375 & -0.112273 & 4.7323 & 0.374297 \\ 0.198319 & 0.112939 & 0.000375 & 4.844827 & 0.486824 \\ -4.646187 & -4.731707 & -4.844241 & 0.000375 & -4.357768 \\ -0.288184 & -0.373631 & -0.486238 & 4.358434 & 0.000375 \end{bmatrix}.$$

As can be seen from the tropical powers, the arbitrage on three currencies is much smaller than the arbitrage on five currencies. The following example is another case in which the longer cycle arbitrages were significantly higher than the three cycle.

Example 5.2:

$$A_2 = \begin{bmatrix} 0 & -0.085911 & -0.19868 & 4.646175 & 0.288121 \\ 0.085911 & 0 & -0.112544 & 4.731885 & 0.37393 \\ 0.19868 & 0.112544 & 0 & 4.844643 & 0.486683 \\ -4.646175 & -4.731885 & -4.844643 & 0 & -4.358161 \\ -0.288121 & -0.37393 & -0.486683 & 4.358161 & 0 \end{bmatrix};$$

$$A_2^{\otimes 3} = \begin{bmatrix} 0.000225 & -0.085603 & -0.198254 & 4.646282 & 0.288228 \\ 0.086136 & 0.000225 & -0.112544 & 4.732311 & 0.374257 \\ 0.19868 & 0.11297 & 0.000225 & 4.844962 & 0.486801 \\ -4.645749 & -4.731874 & -4.844429 & 0.000214 & -4.357746 \\ -0.287794 & -0.373724 & -0.486268 & 4.358172 & 0.000209 \end{bmatrix};$$

$$A_2^{\otimes 4} = \begin{bmatrix} 0.000426 & -0.085603 & -0.198147 & 4.6464 & 0.288429 \\ 0.086136 & 0.000426 & -0.112319 & 4.732418 & 0.374257 \\ 0.198905 & 0.113077 & 0.000426 & 4.844962 & 0.486908 \\ -4.645749 & -4.73166 & -4.844418 & 0.000426 & -4.357628 \\ -0.287588 & -0.373705 & -0.486268 & 4.358381 & 0.000415 \end{bmatrix};$$

$$A_2^{\otimes 5} = \begin{bmatrix} 0.000533 & -0.085485 & -0.198147 & 4.646601 & 0.288547 \\ 0.086361 & 0.000533 & -0.112118 & 4.732418 & 0.374364 \\ 0.199106 & 0.113077 & 0.000533 & 4.84508 & 0.487109 \\ -4.645738 & -4.731459 & -4.844204 & 0.000533 & -4.357628 \\ -0.287588 & -0.373499 & -0.486249 & 4.358587 & 0.000533 \end{bmatrix}.$$

The following is an example where all arbitrage values were quite low.

Example 5.3:

$$A_3 = \begin{bmatrix} 0 & -0.085259 & -0.19786 & 4.646736 & 0.288695 \\ 0.085259 & 0 & -0.112488 & 4.732034 & 0.373987 \\ 0.19786 & 0.112488 & 0 & 4.844524 & 0.486518 \\ -4.646736 & -4.732034 & -4.844524 & 0 & -4.358129 \\ -0.288695 & -0.373987 & -0.486518 & 4.358129 & 0 \end{bmatrix};$$

$$A_3^{\otimes 3} = \begin{bmatrix} 0.000113 & -0.08521 & -0.1977 & 4.646857 & 0.288771 \\ 0.08538 & 0.000113 & -0.112408 & 4.732159 & 0.374067 \\ 0.197911 & 0.112613 & 0.000123 & 4.844684 & 0.486588 \\ -4.646662 & -4.731923 & -4.844483 & 0.000123 & -4.357969 \\ -0.288535 & -0.373866 & -0.486393 & 4.358129 & 0.000123 \end{bmatrix};$$

$$A_3^{\otimes 4} = \begin{bmatrix} 0.00016 & -0.085146 & -0.197667 & 4.6469 & 0.288818 \\ 0.085452 & 0.000125 & -0.112365 & 4.732196 & 0.37411 \\ 0.197983 & 0.112652 & 0.00016 & 4.844717 & 0.486641 \\ -4.646613 & -4.731911 & -4.844401 & 0.00016 & -4.357936 \\ -0.288533 & -0.373794 & -0.486354 & 4.358252 & 0.00016 \end{bmatrix};$$

$$A_3^{\otimes 5} = \begin{bmatrix} 0.000193 & -0.085099 & -0.197624 & 4.646947 & 0.288855 \\ 0.085495 & 0.000193 & -0.112328 & 4.732239 & 0.374153 \\ 0.19802 & 0.112724 & 0.000193 & 4.84477 & 0.486678 \\ -4.646541 & -4.731872 & -4.844364 & 0.000193 & -4.357883 \\ -0.288484 & -0.373782 & -0.486272 & 4.358289 & 0.000193 \end{bmatrix}.$$

The instance in which the five way arbitrage was highest relative to the lower cycles is given by the following matrix.

Example 5.4:

$$A_4 = \begin{bmatrix} 0 & -0.085772 & -0.198271 & 4.64635 & 0.288199 \\ 0.085772 & 0 & -0.112732 & 4.731907 & 0.373953 \\ 0.198271 & 0.112732 & 0 & 4.844615 & 0.486665 \\ -4.64635 & -4.731907 & -4.844615 & 0 & -4.358154 \\ -0.288199 & -0.373953 & -0.486665 & 4.358154 & 0 \end{bmatrix};$$

$$A_4^{\otimes 3} = \begin{bmatrix} 0.000233 & -0.085533 & -0.198262 & 4.646548 & 0.288414 \\ 0.085772 & 0.000233 & -0.112493 & 4.732125 & 0.374166 \\ 0.198504 & 0.112912 & 0.000233 & 4.844854 & 0.486703 \\ -4.646111 & -4.731883 & -4.844406 & 0.000215 & -4.35793 \\ -0.287981 & -0.373729 & -0.486452 & 4.358169 & 0.000204 \end{bmatrix};$$

$$A_4^{\otimes 4} = \begin{bmatrix} 0.000239 & -0.085359 & -0.198038 & 4.646583 & 0.288432 \\ 0.086005 & 0.000239 & -0.11249 & 4.73232 & 0.374186 \\ 0.198684 & 0.112965 & 0.000239 & 4.844857 & 0.486898 \\ -4.646111 & -4.731674 & -4.844382 & 0.000239 & -4.357741 \\ -0.287957 & -0.37372 & -0.486252 & 4.358369 & 0.000224 \end{bmatrix};$$

$$A_4^{\otimes 5} = \begin{bmatrix} 0.000413 & -0.085306 & -0.198032 & 4.646589 & 0.288627 \\ 0.086011 & 0.000413 & -0.112266 & 4.732355 & 0.374204 \\ 0.198737 & 0.112971 & 0.000413 & 4.845052 & 0.486918 \\ -4.645902 & -4.73165 & -4.844376 & 0.000413 & -4.357717 \\ -0.287948 & -0.37352 & -0.486228 & 4.358393 & 0.000413 \end{bmatrix}.$$

On the other end of the spectrum, consider this matrix with a low three and four cycle, and no maximal five cycle involving distinct currencies.

Example 5.5:

$$A_5 = \begin{bmatrix} 0 & -0.085343 & -0.197882 & 4.646705 & 0.288574 \\ 0.085343 & 0 & -0.112527 & 4.732002 & 0.373999 \\ 0.197882 & 0.112527 & 0 & 4.844559 & 0.486535 \\ -4.646705 & -4.732002 & -4.844559 & 0 & -4.358147 \\ -0.288574 & -0.373999 & -0.486535 & 4.358147 & 0 \end{bmatrix};$$

$$A_5^{\otimes 3} = \begin{bmatrix} 0.000082 & -0.085281 & -0.197824 & 4.646803 & 0.288702 \\ 0.085441 & 0.000144 & -0.112413 & 4.732155 & 0.374008 \\ 0.197977 & 0.11268 & 0.000123 & 4.844682 & 0.486556 \\ -4.646577 & -4.732002 & -4.844529 & 0.000144 & -4.357994 \\ -0.288512 & -0.373855 & -0.486382 & 4.358147 & 0.000144 \end{bmatrix};$$

$$A_5^{\otimes 4} = \begin{bmatrix} 0.000128 & -0.085199 & -0.197756 & 4.646849 & 0.288718 \\ 0.085487 & 0.000153 & -0.112383 & 4.732155 & 0.374143 \\ 0.198023 & 0.11268 & 0.000153 & 4.844703 & 0.486679 \\ -4.646561 & -4.731858 & -4.844415 & 0.000153 & -4.357994 \\ -0.28843 & -0.373855 & -0.486382 & 4.358291 & 0.000153 \end{bmatrix};$$

$$A_5^{\otimes 5} = \begin{bmatrix} 0.000144 & -0.085153 & -0.19771 & 4.646865 & 0.2888 \\ 0.085569 & 0.000153 & -0.112374 & 4.73229 & 0.374152 \\ 0.198105 & 0.112701 & 0.000153 & 4.844826 & 0.486688 \\ -4.646515 & -4.731849 & -4.844385 & 0.000153 & -4.357859 \\ -0.288414 & -0.373711 & -0.486268 & 4.3583 & 0.000153 \end{bmatrix}.$$

This matrix is interesting because it is an example of when the optimal five cycle is simply to traverse the optimal four cycle and stay where you are. We also have an example where the optimal four cycle is to traverse the three cycle and then do nothing for the fourth trade. This is also an example of a matrix where there is almost no arbitrage on any length cycle.

Example 5.6:

$$A_6 = \begin{bmatrix} 0 & -0.085209 & -0.19771 & 4.646878 & 0.288791 \\ 0.085209 & 0 & -0.112526 & 4.732026 & 0.373943 \\ 0.19771 & 0.112526 & 0 & 4.844513 & 0.486537 \\ -4.646878 & -4.732026 & -4.844513 & 0 & -4.35814 \\ -0.288791 & -0.373943 & -0.486537 & 4.35814 & 0 \end{bmatrix};$$

$$A_6^{\otimes 3} = \begin{bmatrix} 0.000075 & -0.085095 & -0.197582 & 4.646967 & 0.288902 \\ 0.085223 & 0.000068 & -0.112426 & 4.732151 & 0.37405 \\ 0.197803 & 0.112651 & 0.000164 & 4.844677 & 0.486537 \\ -4.646767 & -4.731919 & -4.844513 & 0.000164 & -4.357976 \\ -0.288663 & -0.373847 & -0.486373 & 4.358144 & 0.000164 \end{bmatrix};$$

$$A_6^{\otimes 4} = \begin{bmatrix} 0.000128 & -0.085041 & -0.197546 & 4.647042 & 0.288955 \\ 0.085284 & 0.000125 & -0.112362 & 4.73219 & 0.374111 \\ 0.197874 & 0.11269 & 0.000164 & 4.844681 & 0.486701 \\ -4.64671 & -4.731862 & -4.844349 & 0.000164 & -4.357976 \\ -0.288627 & -0.373779 & -0.486369 & 4.358304 & 0.000164 \end{bmatrix};$$

$$A_6^{\otimes 5} = \begin{bmatrix} 0.000168 & -0.084984 & -0.197471 & 4.647095 & 0.288991 \\ 0.085348 & 0.000168 & -0.112323 & 4.732251 & 0.374175 \\ 0.19791 & 0.112758 & 0.000168 & 4.844841 & 0.486701 \\ -4.646639 & -4.731823 & -4.844349 & 0.000168 & -4.357812 \\ -0.28857 & -0.373722 & -0.486209 & 4.358304 & 0.000168 \end{bmatrix}.$$

All of these matrices were instances in which there was no two way arbitrage and there was another property which made the matrix in someway special. The instances in which there was a high five way arbitrage relative to three way and four way suggests that even if there is no three way arbitrage (either because it simply does not exist or because transaction costs would eliminate it), a profitable five way arbitrage could still exist. These instances, however, are likely extremely rare. Even with 40,000 simulations, the percent of arbitrage instances where the the five way arbitrage was at least two times greater than the three way arbitrage was only 1.3%.

Finally, here is an example where the arbitrage on three currencies may have been eliminated by transaction costs, but the arbitrage on five currencies was of a high magnitude.

Example 5.7:

$$A_7 = \begin{bmatrix} 0 & -0.08624 & -0.198427 & 4.645719 & 0.287863 \\ 0.08624 & 0 & -0.112718 & 4.731631 & 0.373965 \\ 0.198427 & 0.112718 & 0 & 4.844671 & 0.486787 \\ -4.645719 & -4.731631 & -4.844671 & 0 & -4.358174 \\ -0.287863 & -0.373965 & -0.486787 & 4.358174 & 0 \end{bmatrix};$$

$$A_7^{\otimes 3} = \begin{bmatrix} 0.000531 & -0.085387 & -0.198427 & 4.646534 & 0.28836 \\ 0.08642 & 0.000531 & -0.112187 & 4.732484 & 0.3746 \\ 0.19928 & 0.11333 & 0.000531 & 4.844961 & 0.487005 \\ -4.645391 & -4.731428 & -4.843818 & 0.000525 & -4.357359 \\ -0.287217 & -0.373457 & -0.485972 & 4.358381 & 0.000508 \end{bmatrix};$$

$$A_7^{\otimes 4} = \begin{bmatrix} 0.000853 & -0.085097 & -0.197896 & 4.646534 & 0.288578 \\ 0.086771 & 0.000853 & -0.112007 & 4.732774 & 0.3746 \\ 0.19957 & 0.11333 & 0.000853 & 4.845202 & 0.487318 \\ -4.645188 & -4.7311 & -4.843818 & 0.000853 & -4.357031 \\ -0.287217 & -0.37325 & -0.485644 & 4.358699 & 0.000815 \end{bmatrix};$$

$$A_7^{\otimes 5} = \begin{bmatrix} 0.001143 & -0.085097 & -0.197574 & 4.646775 & 0.288891 \\ 0.087093 & 0.001143 & -0.111656 & 4.732774 & 0.374818 \\ 0.19957 & 0.113571 & 0.001143 & 4.845524 & 0.48764 \\ -4.64486 & -4.730778 & -4.843615 & 0.001143 & -4.357031 \\ -0.28701 & -0.372926 & -0.485644 & 4.359027 & 0.001143 \end{bmatrix}.$$

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